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DYNAMIC OPTIMIZATION METHODS: THEORY AND ITS APPLICATIONS IN WATER ALLOCATION PROBLEMS

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Abstract

The objective of this paper is to evaluate three dynamic optimization methods i.e., mathematical programming, optimal control theory and dynamic programming, to characterize the optimal conditions in solving the problems of optimal water allocation among different sources, users and over time. The first two methods are calculus-based techniques that require the objective function to be differentiable in the decision variables, while the last one uses a backward induction method. Though those methods use different approaches, the results are not significantly different. For empirical studies, mathematical programming and optimal control methods are easier to be applied due to the availability of general-purpose computer software.

Keywords: mathematical programming, optimal control theory, dynamic programming, optimal water allocation

JEL Codes: C61, Q25, Q57

METODE-METODE OPTIMASI DINAMIS: TEORI DAN APLIKASINYA PADA MASALAH ALOKASI SUMBERDAYA AIR

Abstrak

Paper ini bertujuan untuk mengevaluasi tiga metode optimisasi dinamis, yaitu program matematika (*mathematical programming*), metoda kontrol optimal (*optimal control method*) dan program dinamis (*dynamic programming*), dalam mengidentifikasi solusi optimal guna memecahkan masalah-masalah alokasi optimal sumberdaya antar sumber dan pengguna yang berbeda dan antar waktu. Kedua metode pertama menggunakan pendekatan dasar

kalkulus dengan mengaplikasikan turunan pertama dari fungsi tujuan dan kendala, sedangkan metode terakhir menggunakan pendekatan induksi ke belakang (*backward induction*). Walaupun ketiganya menggunakan pendekatan berbeda, namun hasil yang diperolehnya, secara umum, tidaklah berbeda nyata. Untuk keperluan studi empirik, metode program matematika dan metode kontrol optimal lebih mudah diaplikasikan mengingat software (perangkat lunak) komputernya telah tersedia.

Kata Kunci: program matematika, metode optimal kontrol, program dinamis, alokasi air optimal

Kode JEL: C61, Q25, Q57

1. Introduction

Optimization is one of the most important themes in economic analysis. Many economic problems fall into the category of finding the optimal allocation of scarce resources among competing needs. Natural resource management problems frequently have a dynamic element, in the sense that they involve a sequence of decisions over time. Various mathematical optimization techniques can be applied to solve such problems. In this paper, three dynamic optimization techniques are considered; mathematical programming, optimal control theory and dynamic programming.

Mathematical programming is an optimization framework that can handle the problem of maximizing (minimizing) a given linear or non-linear function subject to a set of linear or non-linear constraints. Optimal control theory is a dynamic optimization technique, developed by Pontryagin *et al.* (1952), which deals with an optimization problem subject to constraints in the form of equations of motion. Many economic problems have used these two methods as a means to derive the optimal decision rules for resource allocation in a multiperiod framework. Unlike these two calculus-based methods, generating the first-order conditions which balance marginal benefits against marginal costs, dynamic programming, developed by Bellman (1957), does not require the function being optimized to be differentiable in the decision variables. The orientation of the dynamic programming approach entails balancing current net benefits against all future net benefits by transforming a complex n-variable optimization problem into n simple one-variable optimization problems (Lippman, 1987).

The objective of this paper is mainly to review the basic principles of dynamic optimization methods, including mathematical programming, optimal control theory and dynamic programming. These techniques are then applied to the problems of optimal groundwater allocation among different water users and over time. This review provides a brief introduction to the methods and explains the basic principles of how each method works, including model formulation, derivation of the optimal conditions and the implications of those conditions for a simple water

allocation problem.

2. Mathematical Programming Method

A mathematical programming model refers to a general optimization problem of maximizing (minimizing) a linear or non-linear function subject to a set of equality and inequality constraints (Chiang, 1984; Leonard and Long, 1992). A mathematical programming problem can be stated as choosing instruments within the opportunity set so as to maximize the objective function (Chiang, 1984). Standard notation of a mathematical programming problem can be written

$$\underset{\mathbf{g}}{\text{Maximize}} F(\mathbf{g}) \quad \text{subject to} \quad \mathbf{g} \in \mathbf{G} \quad [1]$$

where \mathbf{g} is a vector of choice variables, (g_1, g_2, \dots, g_n) ; $F(\mathbf{g})$ is the objective function or criterion function, which is the real-valued function of these variables, $F(g_1, g_2, \dots, g_n)$; and the set G of feasible instrument vectors is called the opportunity set.

The basic economic problem of allocating scarce resources among competing ends can be interpreted as one of a mathematical programming model. A particular resource allocation is represented by the choice of a particular vector of instruments; the scarcity of the resource is represented by the opportunity set, reflecting constraints on the instrument; and competing ends are represented by the objective function, which gives the value attached to each of the alternative allocation.

Mathematical programming problems can be classified into two categories, non-linear and linear. Non-linear programming is a special case of the general mathematical programming model [1]. The problem is choosing non-negative values of n choice variables so as to maximize the objective function subject to inequality constraints.

$$\underset{\mathbf{g}}{\text{Maximize}} F(\mathbf{g}) \quad \text{subject to} \quad \mathbf{h}(\mathbf{g}) \leq \mathbf{b}, \quad \mathbf{g} \geq \mathbf{0} \quad [2]$$

where $F(\mathbf{g})$ is assumed to be a real-valued continuously differentiable function of n choice variables \mathbf{g} . The vector function $\mathbf{h}(\mathbf{g})$ is a representation of m constraint functions, $[h_1(g_1, g_2, \dots, g_n), h_2(g_1, g_2, \dots, g_n), \dots, h_m(g_1, g_2, \dots, g_n)]$, and \mathbf{b} is a vector of m constraint constants (b_1, b_2, \dots, b_m) . Therefore, the m inequality constraints in [2] can be written as

$$h_i(g_1, g_2, \dots, g_n) \leq b_i, \quad i = 1, 2, \dots, m \quad [3]$$

These constraints require that all n choice variables are non-negative. Thus, the problem of non-linear programming can be written as

$$\begin{aligned}
& \underset{g_1, g_2, \dots, g_n}{\text{Maximize}} && F(g_1, g_2, \dots, g_n) \\
& \text{subject to} && \begin{cases} h_i(g_1, g_2, \dots, g_n) \leq b_i, & i = 1, 2, \dots, m \\ g_j \geq 0, & j = 1, 2, \dots, n \end{cases}
\end{aligned} \tag{4}$$

A Linear programming model is almost the same as a non-linear model [2], but the problem is choosing non-negative values of n choice variables so as to maximize a linear form of objective function subject to m linear inequality constraints.

$$\underset{g}{\text{Maximize}} \mathbf{c} \mathbf{g} \quad \text{subject to} \quad \mathbf{A} \mathbf{g} \leq \mathbf{b}, \quad \mathbf{g} \geq \mathbf{0} \tag{5}$$

Where A is given m x n matrix, b is a given m x 1 column vector and c is a given 1 x n row vector. Therefore, the problem of linear programming can be written as

$$\begin{aligned}
& \underset{g_1, g_2, \dots, g_n}{\text{Maximize}} && \sum_{j=1}^n c_j g_j \\
& \text{subject to} && \begin{cases} \sum_{j=1}^n a_j g_j \leq b_i, & i = 1, 2, \dots, m \\ g_j \geq 0, & j = 1, 2, \dots, n \end{cases}
\end{aligned} \tag{6}$$

In a linear programming problem, the first-order condition for a local extremum is that the first partial derivatives of the Lagrangian¹ function with respect to all choice variables and the Lagrange multipliers is zero. In non-linear programming, there exists a similar type of first-order condition, known as the Kuhn-Tucker² conditions, which are the single most important analytical result in non-linear programming which can not be accorded the status of necessary conditions unless a certain proviso is satisfied (Chiang, 1984).

Non-linear programming will be further discussed in this research since most studies of water allocation problems have used this method instead of the linear one. The non-linear programming method can be applied to study resource allocation problems in which time plays an important role. Gardner and McGuire (1967), Cummings and Winkelman (1970), Willet and Sharda (1988) and Zachariah (1999) have applied this method to derive the optimal conditions for the problems of dynamic water resource allocation among different categories of water users and water sources over time.

To have an understanding on how this method works in a more realistic

¹ After Lagrange, J. L. 1762. Essai sur une nouvelle méthode pour déterminer les maxima et minima des formules intégrales indéfinies. *Miscellanea Taurinensia* 2:173-195.

² After Kuhn, H. W. and A. W. Tucker (1951) who provided the fundamental contribution in characterizing the nature of the solution to such problems.

problem of water allocation, consider a simple problem of optimal water allocation in which the objective function is the maximization of the present value of net benefits subject to constraints on the stock of groundwater and on the groundwater extraction condition.

$$\text{Maximize}_{g(t)} \quad \Pi = \sum_{t=0}^T \beta^t \pi(S(t), g(t)) dt \quad [7]$$

Subject to:

$$S(t+1) = f(S(t), g(t)) \quad [8]$$

$$h(S(t), g(t), t) \geq b \quad [9]$$

$$S(0) = S_0 \text{ (externally determined)} \quad [10]$$

Equation [7] represents the objective function of the problem, which is stated as a maximization of the present value of net benefits Π , generated from the use of the groundwater resource over a planning period $[0, T]^3$. $\pi(\cdot)$ is the annual net benefit function to the groundwater operator; $S(t)$ represents the stock of groundwater in the aquifer at time t (the state variable of the system); $g(t)$ is the groundwater extraction rate at time t ; $f(\cdot)$ is a function which reflects changes in the groundwater stock variable (state variable) with respect to time; $S(0)$ is the initial level of groundwater stock; $h(\cdot)$ is an inequality constraint; b is a vector of constants; and β^t is a discount factor at year t , where $\beta^t = e^{-rt}$, r is an applicable rate of interest. The values of T and S_0 are exogenously specified.

The Lagrangian function for the above problem is as follows

$$L = \sum_{t=0}^T \beta^t \pi(S(t), g(t)) + \mu_1(t) [S(t+1) - f(S(t), g(t))] + \mu_2(t) [b - h(S(t), g(t), t)] \quad [11]$$

where $\mu_1(t)$ and $\mu_2(t)$ are vectors of Lagrange multipliers associated with constraints [8] and [9]. The Kuhn-Tucker conditions provide a characterization of the solution to the problem represented in the Lagrangian function [11]. The conditions are as follows:

$$\frac{\partial L}{\partial g(t)} = \beta^t \frac{\partial \pi}{\partial g(t)} - \mu_1(t) \frac{\partial f}{\partial g(t)} - \mu_2(t) \frac{\partial h}{\partial g(t)} \leq 0; \quad g(t) \geq 0, \quad g(t) \frac{\partial L}{\partial g(t)} = 0 \quad [12]$$

³ $[,]$ means a closed interval between the two numbers.

$$\frac{\partial L}{\partial S(t)} = \beta' \frac{\partial \pi}{\partial S(t)} - \mu_1(t) \frac{\partial f}{\partial S(t)} - \mu_2(t) \frac{\partial h}{\partial S(t)} \leq 0; \quad S(t) \geq 0, \quad S(t) \frac{\partial L}{\partial S(t)} = 0 \quad [13]$$

$$\frac{\partial L}{\partial \mu_1(t)} = S(t+1) - f(S(t), g(t)) = 0 \quad [14]$$

$$\frac{\partial L}{\partial \mu_2(t)} = b - h(S(t), g(t)) \geq 0; \quad \mu_2 \geq 0, \quad \mu_2(t) \frac{\partial L}{\partial \mu_2(t)} = 0 \quad [15]$$

Equations [12] and [13] are written as inequalities because of the non-negativity restrictions on $g(t)$ and $S(t)$, which allow for the possibility of a boundary solution. Equations [12], [13] and [15] represent complementary slackness conditions of non-linear programming. Equation [12], for example, requires that the present value of the marginal benefit of water be no greater than its aggregate marginal imputed costs. Complementary slackness means that if the optimal solution calls for the active use of groundwater [$g(t) > 0$], the marginal net benefit must be exactly equal to the aggregate imputed marginal costs, as would be the situation in the classical optimization problem. In equation [15], complementary slackness requires that when the inequality constraint is satisfied at the solution as a strict inequality then the corresponding (dual) vector of variables $\mu_2(t)$ equals zero at the solution. These equations [12] to [15] can be used to solve for $g^*(t)$, $S^*(t)$, $\mu_1^*(t)$ and $\mu_2^*(t)$ i.e., optimal rate of groundwater extraction at any time t , level of remaining groundwater stock, and the shadow value associated with each constraint.

3. Optimal Control Theory

3.1 Basic Model

Optimal control theory is a dynamic optimization technique developed by Pontryagin *et al.* (1952) which deals with an optimization problem subject to constraints in the forms of the equation of motion. This method is an improvement over the classical calculus of variations method developed by the Bernoulli brothers in the 1600's which was applied in economics in the late 1920s (Ramsey, 1928 and Hotelling, 1931). Compared to calculus of variations, the optimal control theory can afford insights into a problem that might be less readily apparent through the calculus of variations and also apply to the problems for which the latter is not convenient (Kamien and Schwartz, 1981; Hoy *et al.*, 1996).

In optimal control problems, variables are divided into three categories, the time variable, state variables and control variables. State variables represent those which describe the condition of the system such as the groundwater stock in the aquifer. The rate of change of a state variable may depend on the value of that variable, time or other variables which can be controlled at any time by the operator

of the system. These other variables are called control variables, for instance the groundwater extraction rate. Chiang (1992) stated that it is the control variable that gives the optimal control theory its name and occupies the center of the stage in this approach. The movement of the state variables is governed by first-order differential equations or equations of motion.

A simple optimal control problem for groundwater resource allocation can be formulated as follows (see Syaukat, 2000):

$$\underset{g(t)}{\text{Maximize}} \quad \Pi = \int_0^T \beta^t \pi(S(t), g(t)) dt \quad [16]$$

subject to

$$\dot{S} = f(S(t), g(t), t) \quad [17]$$

$$S(0) = S_0 \text{ (externally determined)} \quad [18]$$

Equation [16] represents the objective function of the problem, which is stated as a maximization of the present value of net benefits, Π , generated from the use of groundwater resource over a planning period $[0, T]$. $\pi(\cdot)$ is the annual net benefit function to the operator; $S(t)$ is the state variable of the system, which is the stock of groundwater in the aquifer at time t ; $g(t)$ is the control variable which is the groundwater extraction rate at time t ; $f(\cdot)$ is the equation of motion (or differential equation) for the state variable; $\dot{S} = \partial S / \partial t$ represents the rate of change of the stock variable (groundwater stock) with respect to time; $S(0)$ and β remain as defined before.

The problem is to choose the path for the control variable $g(t)$ over $[0, T]$ to yield the largest possible Π . To solve the above problem, we define a new function called a Hamiltonian⁴. The Hamiltonian function associated with equations [16] to [18] is

$$H(S(t), g(t), \lambda(t), t) = \beta^t \pi(S(t), g(t), t) + \lambda(t) f(S(t), g(t), t) \quad [19]$$

where $S(t)$, $g(t)$, $f(\cdot)$ and $\pi(\cdot)$ have the same meanings as before, $H(\cdot)$ is the (present value) Hamiltonian function and $\lambda(t)$ is a *costate* variable, which tells how the maximum value varies as the constraint varies. It represents the shadow value of the groundwater resource; the extra value obtainable when a unit of groundwater resource is added.

The maximum principle is applied to solve problem [19]. An optimal solution to the above problem is a triplet, $S(t)$, $g(t)$ and $\lambda(t)$, and must satisfy the following conditions:

⁴ After the Irish mathematician William Rowan Hamilton (1805-1865).

a) The control variable $g(t)$ maximizes $H(S(t),g(t),\lambda(t),t)$ that is,

$$\frac{\partial H}{\partial g(t)} = 0 \Rightarrow \beta^t \frac{\partial \pi}{\partial g(t)} + \lambda(t) \frac{\partial f}{\partial g(t)} = 0 \quad [20]$$

b) The state and costate variables satisfy a pair of differential equations

$$\dot{S}(t) = \frac{\partial H}{\partial \lambda(t)} \Rightarrow \dot{S}(t) = f(S(t), g(t), t) \quad [21]$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial S(t)} \Rightarrow \dot{\lambda}(t) = -\beta^t \frac{\partial \pi}{\partial S(t)} - \lambda(t) \frac{\partial f}{\partial S(t)} \quad [22]$$

c) Satisfy the boundary conditions explained in equation [18].

These three equations, [20], [21] and [22], are the optimal solution to the problem. From the above three equations (one algebraic equation and two differential equations), a specific solution can be derived. Once the optimal control variable path $g^*(t)$ is found, the equation of motion would make it possible to construct the related optimal state variable path $S^*(t)$ and the associated costate variable $\lambda^*(t)$.

3.2 Extended Model

In most cases, the problem of optimal control also includes not only equality constraints, but inequality constraints. In this situation, there are no significant differences in the maximum principle, as those presented in the preceding section. However, when inequality constraints are present, the Hamiltonian function needs to be extended to include the constraints and forms a Lagrangian function. For simplicity, let's consider the above problem with an additional inequality constraint [25].

$$\underset{g(t)}{\text{Maximize}} \quad \Pi = \int_0^T \beta^t \pi(S(t), g(t)) dt \quad [23]$$

Subject to:

$$\dot{S} = f(S(t), g(t), t) \quad [24]$$

$$h(S(t), g(t), t) \geq 0 \quad [25]$$

$$S(0) = S_0 \quad (\text{externally determined}) \quad [26]$$

where $h(\cdot)$ is an inequality constraint, a constant greater than or equal to zero⁵, and included into the problem. The Lagrangian function is the Hamiltonian function [19] plus the constraint [25] multiplied by its associated Lagrange multiplier $\mu(t)$. The Lagrangian function is presented as

$$\begin{aligned} L(S(t), g(t), \lambda(t), \mu(t), t) \\ &= H(S(t), g(t), \lambda(t), t) + \mu(t) h(S(t), g(t), t) \\ &= \beta' \pi(S(t), g(t), t) + \lambda(t) f(S(t), g(t), t) + \mu(t) h(S(t), g(t), t) \end{aligned} \quad [27]$$

As in [27], the Lagrangian function is an extended version of the Hamiltonian function [19]. In this situation, the extended version of the optimal control problem [27] can be solved by using a modified version of the non-linear programming technique thus, the Kuhn-Tucker condition can be applied to solve this problem. The Kuhn-Tucker conditions for the above problem are as follows:

- a) At any time t , the optimal control variable $g^*(t)$ maximizes the Hamiltonian [19] subject to the condition that $g^*(t)$ belongs to the set of admissible controls defined by [29].

$$\frac{\partial L^*}{\partial g(t)} = 0 \Rightarrow \beta' \frac{\partial \pi}{\partial g^*(t)} + \lambda(t) \frac{\partial f}{\partial g^*(t)} + \mu(t) \frac{\partial h}{\partial g^*(t)} = 0 \quad [28]$$

$$\mu(t) \geq 0; \quad h(S^*(t), g^*(t), t) \geq 0; \quad \mu(t) h(S^*(t), g^*(t), t) = 0 \quad [29]$$

where the asterisk on L indicates that the derivatives are evaluated at $(S^*(t), g^*(t))$. The multiplier $\mu(t)$ is piecewise-continuous and continuous on each point of continuity of $g^*(t)$.

- b) The state variable satisfies the differential equation:

$$\dot{S}(t) = \frac{\partial L}{\partial \lambda(t)} \Rightarrow \dot{S}(t) = f(S(t), g(t), t) \quad [30]$$

- c) The costate variable is continuous and has a piecewise-continuous derivative satisfying:

$$\dot{\lambda}(t) = -\frac{\partial L^*}{\partial S(t)} \Rightarrow \dot{\lambda}(t) = -\beta' \frac{\partial \pi}{\partial S(t)} - \lambda(t) \frac{\partial f}{\partial S(t)} - \mu(t) \frac{\partial h}{\partial S(t)} \quad [31]$$

- c) Satisfy the boundary conditions [26].

⁵ Notice that in previous sample, equation [3], the inequality constraint is to be greater than or equal to b , where $b \geq 0$. In this problem, for simplicity, the constraint is written directly to be greater than or equal to 0, and not b .

Similarly, specified solutions for $g^*(t)$, $S^*(t)$, $\lambda^*(t)$ and $\mu^*(t)$ can be found from the above conditions.

4. Dynamic Programming

The basic concept of dynamic programming is the principle of optimality formulated by Bellman (1957):

“an optimal policy has the property that, whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision” (p.83).

Bellman’s principle of optimality allows the optimal solution of the larger problem to be obtained from the solution of a series of smaller ones. A policy is a rule which specifies the decisions to be made as the system passes through the various states. A policy is said to be an optimal policy if the return associated with using it equals the maximal return attainable (Nimhauser, 1966).

There are five key components in dynamic programming: stage, state variables, control (decision) variable, recursive relation (transition function) and return variables. If the term stage refers to an annual period, the stage n is equivalent to year t , which is used in this study. Panel (a) of Figure 1 shows the relationship between these variables at any stage n . The n^{th} stage of the problem requires two input variables; the state variable S_n and the decision variable g_n . The state variable connects the present stage and the previous stage ($n-1$) and allows computation of the remaining amount of the resource. It is also used, in conjunction with the decision variable g_n , to determine the outputs from the stage. There are two outputs from this stage; the return at stage n (Π_n) and the state variable for the next stage $n-1$ (S_{n-1}). The return at any stage is the contribution to the objective function due to the decision and state variables at that stage. The state variables in successive stages are tied together through the recursive relationship that computes the values of S_{n-1} using the values of S_n and g_n .

Panel (b) of Figure 1 shows relationships between stage, decision variables, state variables and returns in a serial multistage decision system. The system consists of a set of stages joined together in series so that the output of one stage becomes the input to the next stage. The backward numbering of the stages represents the characteristic of dynamic programming using the backward induction method which solves the last period first.

4.1 Discrete Time Problem

To look at how dynamic programming works in solving a problem, consider

the same problem of water resource allocation presented in the previous section, with a slight modification of both the objective function and constraints, to represent a discrete time problem. The problem is formulated as follows:

$$\underset{g(t)}{\text{Maximize}} \quad \Pi = \sum_{t=1}^T \beta^t \pi(S(t), g(t)) \quad [32]$$

Subject to:

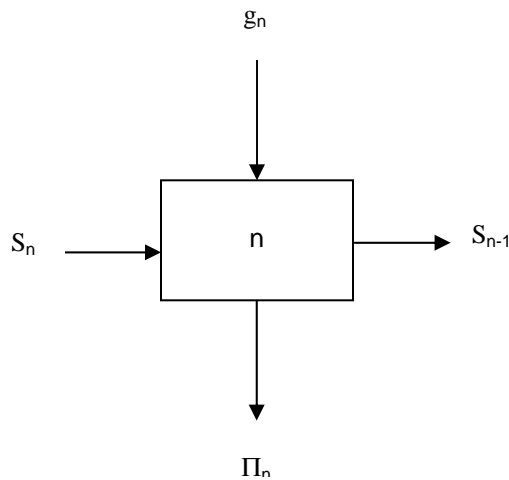
$$S(t+1) = f_t(S(t), g(t), t) \quad [33]$$

$$S(1) = S_1, \quad S(T+1) = \bar{S} \quad (\text{both are externally determined}) \quad [34]$$

The usual dynamic programming terminology is as follows: $\pi(\cdot)$ is the net benefit at time t and $f(\cdot)$ is called the recursive relationship or transition function at t . In this model, the functions $\pi(\cdot)$ and $f(\cdot)$ depend on t and the current values of the state and control variables, $S(t)$ and $g(t)$, but not on their past or future values; and the maximand $\Pi(S(t), g(t))$, which is called the return function, which is the sum of the net benefit functions $\pi(\cdot)$.

In relation to the nature of the optimal solution of problem [32] subject to [33] and [34], the principles of optimality state that an optimal policy has the property that at any stage t , the remaining decision $g^*(t), g^*(t+1), \dots, g^*(T)$ must be optimal with regard to the current state $S^*(t)$, which results from the initial state S_1 , and the earlier decisions $g^*(1), g^*(2), \dots, g^*(t-1)$.

Panel (a). A one-stage decision system



Panel (b). Serial multistage decision system

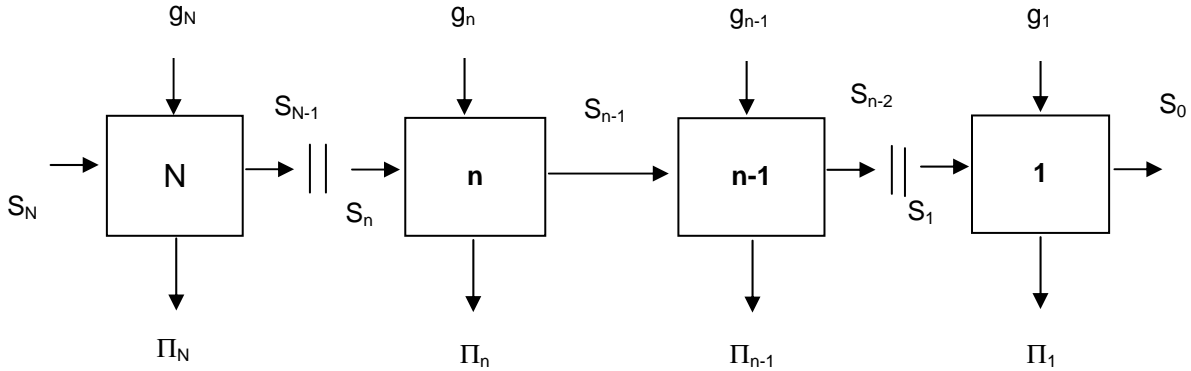


Figure 1. A One-Stage and Serial Multistage Decision System in Dynamic Programming

Notes: g_n is decision variable at stage n
 S_n is state variable at stage n
 Π_n is return of stage n

Source: Nemhauser (1966).

Bellman's principle of optimality gives rise to an important equation called the functional recurrence equation, which is the key to the dynamic programming method of solution. The principle of optimality associated with the above problem is as follows

$$\Pi_t(S(t)) = \underset{g(t)}{\text{Max.}} \beta^t [\pi_t(S(t), g(t)) + \pi_{t+1}(S(t+1))] \quad [35]$$

subject to

$$S(t+1) = f_t(S(t), g(t), t) \quad [36]$$

$$S(1) = S_1, \quad S(T+1) = \bar{S} \quad [37]$$

Combining [35] and [36] results in the following Bellman's functional recurrence equation:

$$\Pi_t(S(t)) = \underset{g(t)}{\text{Max.}} \beta^t [\pi_t(S(t), g(t)) + \pi_{t+1}(f_t(S(t), g(t)))] \quad [38]$$

This equation provides the basis for an efficient method of solution, backward induction. The backward induction method consists of solving the last period first, taking as given the value of the state variable and working backward until the first period, when the value of the state variable is known. The formal procedure of the method is as follows:

- a) At the time T, for given S(T), we choose $g^*(T)$ that solves the problem facing the planner when there is only “one period to go”.

$$\underset{g(T)}{\text{Max.}} \Pi_T = \beta^t \pi_T(S(T), g(T)) \quad [39]$$

Subject to:

$$S(T+1) = f_T(S(T), g(T)) \quad [40]$$

$$S(1) = S_1, \quad S(T+1) = \bar{S} \quad (\text{both are fixed}) \quad [41]$$

where Π_T is the return function for stage (year) T. This problem yields $g^*(T)$ as a function of S(T), $g_T(S(T))$. By definition, $\Pi_T(S(T))$ is the optimal value of the return function for S(T) which was given. Therefore,

$$\Pi_T(S(T)) = \beta^t \pi_T(S(T), g_T(S(T))) \quad [42]$$

- b) Working backward, at time T-1 we seek $g^*(T-1)$ that solves the problem facing the planner when there are “two periods to go”:

$$\begin{aligned} \underset{g(T)}{\text{Max.}} \Pi_{T-1} &= \beta^t \left[\pi_{T-1}(S(T-1), g(T-1)) + \pi_T(S(T)) \right] \\ &= \beta^t \left[\pi_{T-1}(S(T-1), g(T-1)) + \pi_T(S(T), g_T(S(T))) \right] \end{aligned} \quad [43]$$

Subject to

$$S(T) = f_{T-1}(S(T-1), g(T-1)); \quad S(T-1) \text{ given} \quad [44]$$

This gives $g^*(T-1)$ as function of S(T-1), $g^*(T-1) = g_{T-1}(S(T-1))$. This equation means that the optimal rate of groundwater extraction at year T-1 is a function of the groundwater stock at that period. The optimal value of Π_{T-1} is obtained by substituting [42] and [44] into [43]

$$\Pi_{T-1}(S(T-1)) = \beta^t \left[\begin{array}{l} \pi_{T-1}\{(S(T-1), g(T-1)), g_{T-1}(S(T-1))\} + \\ \left. \pi_T \left\{ \begin{array}{l} f_{T-1}[S(T-1), g_{T-1}(S(T-1))], \\ g_T[f_{T-1}(S(T-1), g_{T-1}(S(T-1)))] \end{array} \right\} \right] \end{array} \right] \quad [45]$$

which is a composite of the known function and has S(T-1) for sole argument. This process is repeated until t=1 is reached. In general, the solution of the optimal decision variable at any stage t is a function of the state variable in the same stage

$$g^*(t) = h(S(t)) \quad [46]$$

where $g^*(t)$ is the optimal decision variable i.e., optimal rate of groundwater extraction, at stage (year) t and S(t) is the state variable at that period i.e., stock of groundwater.

4.2 Continuous Time Problem

Dynamic programming applies different procedures for solving problems with continuous time. The continuous time problem is given as

$$\underset{g(t)}{\text{Maximize}} \quad \Pi = \int_0^T \beta^t \pi(S(t), g(t)) dt \quad [47]$$

Subject to:

$$\dot{S} = f(S(t), g(t), t) \quad [48]$$

$$S(0) = S_0, \quad S(T) = S_T \quad (\text{both are externally determined}) \quad [49]$$

where $\pi(\cdot)$ is the net benefit at time t and $f(\cdot)$ is called the transition function at t, S(t) and g(t) are the state and control variables, and $\Pi(S(t), g(t))$ is the return function.

In a continuous-time dynamic programming problem, the Hamiltonian-Jacobi-Bellman equation is used, rather than the functional of recurrence equation, which provides an alternative method to the optimal control theory for solving continuous time control (Leonard and Long, 1992). Applying the principle of optimality, the problem can be presented as

$$\Pi(S(t), t) = \underset{g(t)}{\text{Max.}} \left[\int_t^{t+\Delta t} \beta^t \pi(S(t), g(t)) dt + \pi(S(t+\Delta t), t+\Delta t) \right] \quad [50]$$

For a sufficiently small Δt , equation [50] can be rewritten as

$$\Pi(S(t), t) = \underset{g(t)}{\text{Max.}} \left[\beta^t \pi(S(t), g(t)) \Delta t + \Pi(S(t + \Delta t), t + \Delta t) + O(\Delta t) \right] \quad [51]$$

where $O(\Delta t)$ is the sum of higher-order terms in Δt , resulting from the Taylor's expansion. Assuming that Π is continuously differentiable, [51] can be written as

$$\begin{aligned} \Pi(S(t + \Delta t), t + \Delta t) &= \left[\beta^t \Pi(S(t), g(t)) + \Pi_s \Delta S + \Pi_t \Delta t + O(\Delta t) \right] \\ &= \left[\beta^t \Pi(S(t), g(t)) + \left(\Pi_s \dot{S} + \Pi_t \right) \Delta t + O(\Delta t) \right] \end{aligned} \quad [52]$$

where Π_s and Π_t are differentiations of Π with respect to the state variable and time. Equation [52] indicates that the net benefits associated with small changes in the state variable, due to a small change in time, are equal to the sum of the present value of the initial net benefits, at time t , plus the marginal benefits associated with those changes plus the sum of higher order terms. Substituting [52] into [51] results in the following

$$\Pi(S(t), t) = \underset{g(t)}{\text{Max.}} \left[\begin{aligned} &\beta^t \Pi(S(t), g(t), t) \Delta t + \Pi(S(t), t) \\ &+ \Pi_s f(S(t), g(t), t) \Delta t + \Pi_t \Delta t \end{aligned} \right] + O(\Delta t) \quad [53]$$

Cancel $\Pi(S(t), g(t), t)$ on both sides and divide the resulting equation by Δt , taking the limit $\Delta t \rightarrow 0$ results in the following:

$$0 = \underset{g(t)}{\text{Max.}} \left[\beta^t \pi(S(t), g(t)) + \Pi_s f(S(t), g(t), t) + \Pi_t \right] \quad [54]$$

This is the so called Hamiltonian-Jacobi-Bellman equation. If the Hamiltonian is defined as the one of equation [19]:

$$H(S(t), g(t), \lambda(t), t) = \beta^t \pi(S(t), g(t), t) + \lambda(t) f(S(t), g(t), t)$$

where $\lambda(t) = \Pi_s(S(t), t)$ i.e., marginal value associated with changes in the state variable. Therefore, [54] can be written as

$$\underset{g(t)}{\text{Max.}} H(.) = -\Pi_t \quad [55]$$

Equation [55] or [54] is a partial differential equation since it involves the partial derivatives of Π with respect to S and t . However, in general, this type of equation is difficult to solve, even for very simple π and f functions (Leonard and Long, 1992).

5. Conclusions

Mathematical programming, particularly the non-linear programming method, has important applications in economic theory, leading to conditions characterizing an equilibrium at an optimum point. Non-linear programming can be used to solve dynamic or intertemporal resource allocation problems, particularly for discrete time problems. The main advantage of using non-linear programming is that it can include more choice variables and state variables compared to optimal control and dynamic programming. This is an important factor since real world water allocation problems generally include a number of choice and state variables. The more activities that are considered, the more variables that will be involved. In addition, various computational approaches have been developed to solve non-linear programming problems, and such approaches are widely available and routinely used to solve particular problems (Murtagh and Sanders, 1977; Intriligator, 1987).

Optimal control theory has also been used by economists to study problems involving optimal decisions in a multiperiod framework. As described in the application sections, the optimal control method has offered a technique for simultaneously determining optimal spatial and intertemporal allocation of the water resource system, consistent with economic and hydrological theory and based on the water values in current and in future use. The optimal control method offers a technique which provides the water manager with a specification of the hydrological system and additional economic information, including direct calculation of the marginal user cost or stock value of groundwater and surface water.

The disadvantages of the optimal control method take the form of a limited number of state and control variables that can be specified, particularly for the problem with complex differential equations. In addition, optimal control alone can not be used for the problems which involve inequality constraints. In this situation, Hamiltonian must be extended to form the Lagrangian function and use the Kuhn-Tucker method to derive the optimality conditions.

The extended-version of optimal control and non-linear mathematical programming problems, therefore, use the same procedures in deriving the optimal condition. Dorfman (1969) compares the optimal conditions resulting from both Hamiltonian and Lagrangian functions applied to the same problem. He argues that the basic equations of the maximum principle are seen to be the limiting forms of the first-order necessary conditions of the non-linear programming for an optimum applied to the same problem; and the costate variables of the maximum principles are the limiting values of the Lagrange multipliers.

Dynamic programming is a non calculus-based technique. The orientation of the typical dynamic programming approach entails balancing current benefits against all future benefits by transforming a complex n -variable optimization problem into n simple one-variable optimization problem. The primary attention of the dynamic programming method is focussed on the optimal value of the objective function, rather than on the properties of the optimal control and state paths. The dynamic programming technique solves a problem by transforming a sequential

decision process containing many interdependent variables into a series of single-stage problems, each containing only a few variables. From this process, then, the objective function is computed using backward recursions. Under these circumstances, the characteristics of the solution are difficult to identify. In fact, the optimal value function is used as a characterization of the solution.

The difficulties with this approach are usually in the formulation of the functional (recursive) equation. Second, the problem is not solved until the structure of an optimal policy is exhibited, in which it is often the case that there is more than one optimal policy (Lippman,1987). Third, dynamic programming has disadvantages associated with a severe limitation of the number of state and control variables that can be specified. Noel and Howitt (1982) argue that the use of dynamic programming will restrict the specification of the groundwater and surface water systems to low dimension representations of the system.

6. References

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