

2



ISSN 0972-0871



Reprinted from the

Far East Journal of
Mathematical Sciences (FJMS)

Volume 82 Number 1, 2016, pp 33-44

ESTIMATING THE INTENSITY OBTAINED AS THE
PRODUCT OF A PERIODIC FUNCTION WITH THE
QUADRATIC TREND OF A NONHOMOGENEOUS
POISSON PROCESS

W. W. Mungku, R. B. Bhatia

Hastinapur, Gwalior



Poulage Poulage Publishing House
Village Nawal, 452001, India
Allahabad-201002, INDIA
http://qp.org.in/journal/ Far East Journal of
Mathematical Sciences



Information for Authors

Aims and Scope: The Far East Journal of Mathematical Sciences (FJMS) is devoted to publishing original research papers and critical survey articles in the field of Pure and Applied Mathematics, Computer Applications and Statistics. The FJMS is a fortnightly journal published in twelve volumes annually and each volume comprises of two issues.

Abstracting, Indexing and Reviews: Global Impact Factor : 0.692, Scopus, Mathematical Reviews, MathSciNet, ProQuest, IndexCopernicus, EBSCOhost, Zentralblatt MATH, Ulrich's web, Indian Science Abstracts, SCIRUS, OCLC, Google Scholar, Excellence in Research for Australia (ERA), AcademicKeys.

Submission of Manuscripts: Authors may submit their papers for consideration in the Far East Journal of Mathematical Sciences (FJMS) by the following modes:

1. **Online submission:** Please visit journal's homepage at <http://www.pphmj.com/journals/fjms.htm>
2. **Electronically:** At the e-mail address: fjms@pphmj.com or kkazad@pphmj.com
3. **Hard copies:** Papers in duplicate with a letter of submission at the address of the publisher.

The paper must be typed only on one side in double spacing with a generous margin all round. An effort is made to publish a paper duly recommended by a referee within a period of three months. One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections.

Abstract and References: Authors are requested to provide an abstract of not more than 250 words and latest Mathematics Subject Classification. Statements of Lemmas, Propositions and Theorems should be set in *italics* and references should be arranged in alphabetical order by the surname of the first author.

Page Charges and Reprints: Authors are requested to arrange page charges of their papers @ USD 40.00 per page for USA and Canada, and EUR 30.00 per page for rest of the world from their institutions/research grants, if any. However, for authors in India this charge is Rs. 800.00 per page. No extra charges for colour figures. Twenty-five reprints in print version and a copy in soft version are provided to the corresponding author ex-gratis. Additional sets of reprints may be ordered at the time of proof correction.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the Pushpa Publishing House, Allahabad, India, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Subscription Information for 2013

Institutional Price for all countries except India

Electronic Subscription	€ 720.00	US\$ 995.00
Print Subscription includes Online Access	€ 1195.00	US\$ 1695.00

For Institutions: On seeking a license for volume(s) of the Far East Journal of Mathematical Sciences (FJMS), the facility to download and print the articles will be available through the institutional IP address to be provided by the appropriate authority. The facility to download will continue till the end of the next calendar year from the last issue of the volume subscribed. For having continued facility to keep the download of the same subscribed volume for another two calendar years may be had on a considerable discounted rate.

Price in Indian Rs. (For Indian Institutions in India only)

Print Subscription Only	Rs. 15000.00
-------------------------	--------------

The subscription year runs from January 1, 2013 through December 31, 2013.

Information: The journals published by the "Pushpa Publishing House" are solely distributed by the "Vijaya Books and Journals Distributors".

Contact Person: Subscription Manager, Vijaya Books and Journals Distributors, Vijaya Niwas, 198 Mumfordganj, Allahabad 211002, India; sub@pphmj.com; arun@pphmj.com

Information for Special Volume 2013

Special Volume 2013 (supplementary volume) of the Far East Journal of Mathematical Sciences (FJMS) is devoted to articles on Computer Sciences, Information Sciences, Financial Management and Biological Sciences, consisting of six issues to appear in the month of February, April, June, August, October and December. It is in addition to its regular volumes.

The special volume considers original research papers and critical survey articles basically lying in the following areas:

1. Development of theory and methods for Computer and Information Sciences
2. Intrinsic force bringing Mathematics and Computer Science closure for scientific and engineering advancement
3. Impact of Mathematical techniques in Biological Sciences
4. Application of Mathematics in Financial Management
5. Analysis of algorithms and Software tools for computational work

Institutional Price for all countries except India

Electronic Subscription	€ 215.00	US\$ 315.00
Print Subscription includes Online Access	€ 355.00	US\$ 525.00

Price in Indian Rs. (For Indian Institutions in India only)

Print Subscription Only	Rs. 3500.00
-------------------------	-------------

ESTIMATING THE INTENSITY OBTAINED AS THE PRODUCT OF A PERIODIC FUNCTION WITH THE QUADRATIC TREND OF A NON-HOMOGENEOUS POISSON PROCESS

I W. Mangku¹, R. Budiarti¹, Taslim² and Casman³

¹Department of Mathematics
 Bogor Agricultural University
 Jl. Meranti, Kampus IPB Darmaga
 Bogor 16680, Indonesia
 e-mail: wayan.mangku@gmail.com

²Madrasah Aliyah Negeri 1
 Lubuklinggau, Bengkulu, Indonesia

³Madrasah Tsanawiyah Negeri
 Jatibarang, Indramayu, Indonesia

Abstract

A kernel-type nonparametric estimator of the intensity obtained as the product of a periodic function with the quadratic trend of a non-homogeneous Poisson process is constructed and investigated. It is considered the case when there is only a single realization of the Poisson process is observed in a bounded interval. The proposed estimator is proved to be weakly and strongly consistent when the size of the interval indefinitely expands. The asymptotic bias, variance, and the mean-squared error of the proposed estimator are also computed.

Received: July 17, 2013; Accepted: August 7, 2013

2010 Mathematics Subject Classification: 62E20, 62G05, 62G20.

Keywords and phrases: Poisson process, periodic intensity function, quadratic trend, kernel-type estimator, consistent estimation, bias, variance, mean-squared error.

1. Introduction

We consider a non-homogeneous Poisson process N on $[0, \infty)$ with (unknown) locally integrable intensity function λ . The intensity function λ is assumed to be a product of a periodic function with the quadratic trend. That is, for any given point $s \in [0, \infty)$, the intensity function λ can be written as

$$\lambda(s) = (\lambda_c^*(s))(as^2), \tag{1.1}$$

where $\lambda_c^*(s)$ is a periodic function with known period τ and a denotes the slope of the quadratic trend. We do not assume any (parametric) form of λ_c^* except that it is periodic.

Since $a\lambda_c^*$ is also a periodic function with period τ , without loss of generality, the intensity function λ given in (1.1) can also be written as

$$\lambda(s) = (\lambda_c(s))(s^2), \tag{1.2}$$

where $\lambda_c(s) = a\lambda_c^*(s)$. Hence, for each point $s \in [0, \infty)$ and all $k \in \mathbf{Z}$, where \mathbf{Z} denotes the set of integers, we have

$$\lambda_c(s + k\tau) = \lambda_c(s). \tag{1.3}$$

Using (1.2) and (1.3), the problem of estimating λ at a given point $s \in [0, \infty)$ can be reduced to a problem of estimating λ_c at a given point $s \in [0, \tau)$. Hence, for the rest of the paper, we restrict our attention to the problem of estimating λ_c at a given point $s \in [0, \tau)$.

Suppose that, for some $\omega \in \Omega$, a single realization $N(\omega)$ of the Poisson process N defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ given by (1.2) is observed, though only within a bounded interval $[0, n]$. Our objective in this paper is: (a) To construct a kernel-type estimator for λ_c at a given point $s \in [0, \tau)$ using only a single realization $N(\omega)$ of the Poisson

process N observed in interval $[0, n]$. (b) To prove, under the minimal conditions, that our estimator is weakly and strongly consistent in estimating $\lambda_c(s)$. (c) To compute the asymptotic bias, variance, and the mean-squared error of the proposed estimator.

We will assume throughout that s is a Lebesgue point of λ , that is, we have

$$\lim_{h \downarrow 0} \frac{1}{2h} \int_{-h}^h |\lambda(s+x) - \lambda(s)| dx = 0 \tag{1.4}$$

(e.g., see [5, pp. 107-108]), which automatically means that s is a Lebesgue point of λ_c as well. This assumption is a mild one since the set of all Lebesgue points of λ is dense in \mathbf{R} , whenever λ is assumed to be locally integrable.

We refer to [4] for some parallel results when the intensity function λ is assumed to satisfy $\lambda(s) = (\lambda_c(s))(as)$, that is the product of a periodic function with the linear trend. See also [1] and [3] for some results when the intensity function is assumed to satisfy $\lambda(s) = \lambda_c(s) + as$.

2. Construction of the Estimator and Results

Let $K : \mathbf{R} \rightarrow \mathbf{R}$ be a real valued function, called *kernel*, which satisfies the following conditions: (K1) K is a probability density function, (K2) K is bounded, and (K3) K has (closed) support $[-1, 1]$. Let also h_n be a sequence of positive real numbers converging to 0, that is,

$$h_n \downarrow 0, \tag{2.1}$$

as $n \rightarrow \infty$. Now we may define the estimator of λ_c at a given point $s \in [0, \tau)$ as follows:

$$\hat{\lambda}_{c,n,K}(s) = \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n(s+k\tau)^2} \int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right) N(dx). \tag{2.2}$$

Next we describe the idea behind the construction of the kernel-type estimator $\hat{\lambda}_{c,n,K}(s)$ of $\lambda_c(s)$. Since there is available only one realization of the Poisson process N , we have to collect necessary information about the (unknown) value of $\lambda_c(s)$ from different places of the interval $[0, n]$. For this reason, assumption (1.3) plays a crucial role and leads to the following set of (approximate) equations. Let

$$N_n = \#\{k : s + k\tau \in [0, n]\},$$

where $\#$ denotes the number of elements. Then we have

$$\begin{aligned} \lambda_c(s) &= \frac{1}{N_n} \sum_{k=0}^{\infty} \lambda_c(s + k\tau) \mathbf{I}\{s + k\tau \in [0, n]\} \\ &= \frac{1}{N_n} \sum_{k=0}^{\infty} \frac{\lambda(s + k\tau)}{(s + k\tau)^2} \mathbf{I}\{s + k\tau \in [0, n]\} \\ &\approx \frac{1}{N_n} \sum_{k=0}^{\infty} \frac{1}{(s + k\tau)^2} \frac{1}{2h_n} \int_{s+k\tau-h_n}^{s+k\tau+h_n} \lambda(x) \mathbf{I}(x \in [0, n]) dx \\ &= \frac{1}{N_n} \sum_{k=0}^{\infty} \frac{1}{2h_n(s + k\tau)^2} \mathbf{E}N([s + k\tau - h_n, s + k\tau + h_n] \cap [0, n]) \\ &\approx \frac{1}{N_n} \sum_{k=0}^{\infty} \frac{1}{2h_n(s + k\tau)^2} N([s + k\tau - h_n, s + k\tau + h_n] \cap [0, n]) \\ &\approx \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{2h_n(s + k\tau)^2} N([s + k\tau - h_n, s + k\tau + h_n] \cap [0, n]), \end{aligned} \quad (2.3)$$

where \mathbf{I} denotes the indicator function. We note that in order to make the first equality in (2.3) work, we have assumed that s is a Lebesgue point of λ and h_n converges to 0. Thus, from (2.3), we conclude that

$$\hat{\lambda}_{c,n}(s) = \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{2h_n(s + k\tau)^2} N([s + k\tau - h_n, s + k\tau + h_n] \cap [0, n]), \quad (2.4)$$

is an estimator of $\lambda_c(s)$. Note that the estimator $\hat{\lambda}_{c,n}(s)$ can be rewritten as

$$\begin{aligned} \hat{\lambda}_{c,n}(s) &= \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n(s + k\tau)^2} \\ &\quad \times \int_0^n \frac{1}{2} \mathbf{I}_{[-1,1]}([s + k\tau - h_n, s + k\tau + h_n]) N(dx). \end{aligned} \quad (2.5)$$

By replacing the function $\frac{1}{2} \mathbf{I}_{[-1,1]}(\cdot)$ in (2.5) by the general kernel $K(\cdot)$, we immediately arrive at the estimator introduced in (2.2).

Our main results are presented in the following theorems.

Theorem 1 (Consistency). *Suppose that the intensity function λ satisfies (1.2) and is locally integrable. If the kernel K satisfies conditions (K1), (K2), (K3), the bandwidth h_n satisfies assumptions (2.1) and $n^2 h_n \rightarrow \infty$, then*

$$\hat{\lambda}_{c,n,K}(s) \xrightarrow{P} \lambda_c(s), \quad (2.6)$$

as $n \rightarrow \infty$, provided s is a Lebesgue point of λ . In other words, $\hat{\lambda}_{c,n,K}(s)$ is a consistent estimator of $\lambda_c(s)$. In addition, the Mean-Squared Error (MSE) of $\hat{\lambda}_{c,n,K}(s)$ converges to 0, as $n \rightarrow \infty$, that is, we have

$$MSE(\hat{\lambda}_{c,n,K}(s)) \rightarrow 0, \quad (2.7)$$

as $n \rightarrow \infty$.

Under naturally a stronger assumption on the choice of bandwidth h_n , we also have complete convergence of $\hat{\lambda}_{c,n,K}(s)$, which is given in the following theorem.

Theorem 2 (Complete convergence). *Suppose that the intensity function λ satisfies (1.2) and is locally integrable. If the kernel K satisfies conditions (K1), (K2), (K3), the bandwidth $h_n = n^{-\alpha}$ with $0 < \alpha < 1$, then*

$$\hat{\lambda}_{c,n,K}(s) \xrightarrow{c} \lambda_c(s),$$

as $n \rightarrow \infty$, provided s is a Lebesgue point of λ . In other words, $\hat{\lambda}_{c,n,K}(s)$ converges completely to $\lambda_c(s)$ as $n \rightarrow \infty$.

Note also that, by Theorem 2 and the Borel-Cantelli Lemma, we have strong consistency of $\hat{\lambda}_{c,n,K}(s)$, that is, we have

$$\hat{\lambda}_{c,n,K}(s) \xrightarrow{a.s.} \lambda_c(s),$$

as $n \rightarrow \infty$.

Asymptotic approximations to the bias and variance of $\hat{\lambda}_{c,n,K}(s)$ are given in the following two theorems.

Theorem 3 (Asymptotic approximation to the bias). *Suppose that the intensity function λ satisfies (1.2), is locally integrable and λ_c has finite second derivative λ_c'' at s . If the kernel K is symmetric and satisfies conditions (K1), (K2), (K3), h_n satisfies assumptions (2.1) and $nh_n^2 \rightarrow \infty$, then*

$$E\hat{\lambda}_{c,n,K}(s) = \lambda_c(s) + \frac{1}{2}\lambda_c''(s)h_n^2 \int_{-1}^1 x^2 K(x) dx + o(h_n^2), \quad (2.8)$$

as $n \rightarrow \infty$.

Theorem 4 (Asymptotic approximation to the variance). *Suppose that the intensity function λ satisfies (1.2) and is locally integrable. If the kernel K satisfies conditions (K1), (K2), (K3) and h_n satisfies assumptions (2.1), then*

$$Var(\hat{\lambda}_{c,n,K}(s)) = \frac{\pi^2 \lambda(s)}{6n^2 h_n} \int_{-1}^1 K^2(x) dx + o\left(\frac{1}{n^2 h_n}\right) \quad (2.9)$$

as $n \rightarrow \infty$, provided s is a Lebesgue point of λ .

We note in passing that, the r.h.s. of (2.8) is the same as the r.h.s. of (2.9) of [4], the case when the intensity function is assumed to be the product of a periodic function with the linear trend. The same asymptotic approximation also appears in (2.5) of [2], the case when the intensity function is assumed to be purely periodic. However, we have slightly different asymptotic approximation to the variance of the estimator (cf. Theorem 4) compared to the one in [4] (cf. (2.10) of [4]) as well as the one in [2] (cf. (3.4) of [2]).

From Theorems 3 and 4, one can obtain an asymptotic approximation to the Mean-Squared Error (MSE) of $\hat{\lambda}_{c,n,K}(s)$, which is given by

$$MSE(\hat{\lambda}_{c,n,K}(s)) = \frac{\pi^2 \lambda(s)}{6n^2 h_n} \int_{-1}^1 K^2(x) dx + \frac{1}{4} \left(\lambda_c''(s) \int_{-1}^1 x^2 K(x) dx \right)^2 h_n^4 + o\left(\frac{1}{n^2 h_n}\right) + o(h_n^4), \quad (2.10)$$

as $n \rightarrow \infty$. By minimizing the sum of the first and second terms of (2.10) (the main terms for the variance and the squared bias), one can obtain the optimal choice of h_n , which is given by

$$h_n = \left[\frac{\pi^2 \lambda(s) \int_{-1}^1 K^2(x) dx}{\left(\lambda_c''(s) \int_{-1}^1 x^2 K(x) dx \right)^2} \right]^{\frac{1}{5}} n^{-\frac{2}{5}}.$$

With this choice of h_n , the optimal rate of decrease of $MSE(\hat{\lambda}_{c,n,K}(s))$ is of order $O(n^{-8/5})$ as $n \rightarrow \infty$.

3. Proofs

Before proving Theorems 1, 2, 3 and 4, first we state and prove the following lemma, which is needed for proving Theorems 1 and 2.

Lemma 1 (Asymptotic unbiasedness). *Suppose that the intensity function λ satisfies (1.2) and is locally integrable. If the kernel K satisfies conditions (K1), (K2), (K3) and h_n satisfies assumptions (2.1), then*

$$\mathbf{E}\hat{\lambda}_{c,n,K}(s) \rightarrow \lambda_c(s), \quad (3.1)$$

as $n \rightarrow \infty$, provided s is a Lebesgue point of λ .

Proof. The expectation on the l.h.s. of (3.1) can be computed as follows:

$$\begin{aligned} & \mathbf{E}\hat{\lambda}_{c,n,K}(s) \\ &= \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n(s+k\tau)^2} \int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right) \mathbf{E}N(dx) \\ &= \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n(s+k\tau)^2} \int_{\mathbf{R}} K\left(\frac{x-(s+k\tau)}{h_n}\right) \lambda(x) \mathbf{I}(x \in [0, n]) dx. \end{aligned} \quad (3.2)$$

By a change of variable and using (1.2) and (1.3), the r.h.s. of (3.2) can be written as

$$\begin{aligned} & \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n(s+k\tau)^2} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \lambda(x+s+k\tau) \mathbf{I}(x+s+k\tau \in [0, n]) dx \\ &= \frac{\tau}{nh_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \lambda_c(x+s) \sum_{k=0}^{\infty} \frac{(x+s+k\tau)^2}{(s+k\tau)^2} \mathbf{I}(x+s+k\tau \in [0, n]) dx. \end{aligned} \quad (3.3)$$

By noting that

$$\sum_{k=0}^{\infty} \frac{(x+s+k\tau)^2}{(s+k\tau)^2} \mathbf{I}(x+s+k\tau \in [0, n]) = \frac{n}{\tau} + \mathcal{O}(1), \quad (3.4)$$

as $n \rightarrow \infty$ uniformly in $x \in [-h_n, h_n]$, we obtain

$$\mathbf{E}\hat{\lambda}_{c,n,K}(s) = \frac{1}{h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \lambda_c(x+s) dx + \mathcal{O}\left(\frac{1}{n}\right), \quad (3.5)$$

as $n \rightarrow \infty$. The first term on the r.h.s. of (3.5) can be written as

$$\frac{1}{h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) (\lambda_c(x+s) - \lambda_c(s)) dx + \frac{\lambda_c(s)}{h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) dx. \quad (3.6)$$

Since s is a Lebesgue point of λ_c (cf. (1.4)) and the kernel K satisfies conditions (K2) and (K3), it is easily seen that the first term of (3.6) is $o(1)$, as $n \rightarrow \infty$. By the assumption $\int_{\mathbf{R}} K(x) dx = 1$ (cf. (K1)), the second term of (3.6) is equal to $\lambda_c(s)$. Clearly, the second term on the r.h.s. of (3.5) is $o(1)$, as $n \rightarrow \infty$. Hence, the r.h.s. of (3.5) is equal to $\lambda_c(s) + o(1)$, as $n \rightarrow \infty$. This completes the proof of Lemma 1.

Proof of Theorem 3. By (2.1) and Young's form of Taylor's theorem, the first term on the r.h.s. of (3.5) can be written as

$$\begin{aligned} & \frac{1}{h_n} \int_{-h_n}^{h_n} K\left(\frac{x}{h_n}\right) \lambda_c(x+s) dx = \int_{-1}^1 K(x) \lambda_c(s+xh_n) dx \\ &= \lambda_c(s) + \lambda'_c(s) h_n \int_{-1}^1 x K(x) dx + \frac{\lambda''_c(s)}{2} h_n^2 \int_{-1}^1 x^2 K(x) dx + o(h_n^2), \end{aligned}$$

as $n \rightarrow \infty$. Since the kernel K is symmetric around zero, we have that $\int_{-1}^1 x K(x) dx = 0$. By the assumption $nh_n^2 \rightarrow \infty$, the second term on the r.h.s. of (3.5) is $o(h_n^2)$, as $n \rightarrow \infty$. Hence we have (2.8). This completes the proof of Theorem 3.

Proof of Theorem 4. The variance of $\hat{\lambda}_{c,n,K}(s)$ can be computed as follows:

$$\text{Var}(\hat{\lambda}_{c,n,K}(s)) = \frac{\tau^2}{n^2} \text{Var}\left(\sum_{k=0}^{\infty} \frac{1}{h_n(s+k\tau)^2} \int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right) N(dx)\right). \quad (3.7)$$

By (2.1), for sufficiently large n , we have that the intervals $[s + k\tau - h_n, s + k\tau + h_n]$ and $[s + j\tau - h_n, s + j\tau + h_n]$ are not overlap for all $k \neq j$. This implies, for all $k \neq j$,

$$K\left(\frac{x - (s + k\tau)}{h_n}\right)N(dx) \text{ and } K\left(\frac{x - (s + j\tau)}{h_n}\right)N(dx)$$

are independent random variables. Hence, the r.h.s. of (3.7) can be computed as follows:

$$\begin{aligned} & \frac{\tau^2}{n^2 h_n^2} \sum_{k=0}^{\infty} \frac{1}{(s + k\tau)^4} \int_0^n K^2\left(\frac{x - (s + k\tau)}{h_n}\right) \text{Var}(N(dx)) \\ &= \frac{\tau^2}{n^2 h_n^2} \sum_{k=0}^{\infty} \frac{1}{(s + k\tau)^4} \int_0^n K^2\left(\frac{x - (s + k\tau)}{h_n}\right) \mathbf{E}N(dx) \\ &= \frac{\tau^2}{n^2 h_n^2} \sum_{k=0}^{\infty} \frac{1}{(s + k\tau)^4} \int_0^n K^2\left(\frac{x - (s + k\tau)}{h_n}\right) \lambda(x) dx. \end{aligned} \quad (3.8)$$

By a change of variable and using (1.2) and (1.3), the r.h.s. of (3.8) can be written as

$$\begin{aligned} & \frac{\tau^2}{n^2 h_n^2} \sum_{k=0}^{\infty} \frac{1}{(s + k\tau)^4} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right) \lambda(x + s + k\tau) \mathbf{I}(x + s + k\tau \in [0, n]) dx \\ &= \frac{\tau^2}{n^2 h_n^2} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right) \lambda_c(x + s) \sum_{k=0}^{\infty} \frac{(x + s + k\tau)^2}{(s + k\tau)^4} \mathbf{I}(x + s + k\tau \in [0, n]) dx. \end{aligned} \quad (3.9)$$

Now note that

$$\sum_{k=0}^{\infty} \frac{(x + s + k\tau)^2}{(s + k\tau)^4} \mathbf{I}(x + s + k\tau \in [0, n]) = \frac{\pi^2}{6\tau^2} + o(1), \quad (3.10)$$

as $n \rightarrow \infty$ uniformly in $x \in [-h_n, h_n]$. Then, the r.h.s. of (3.9) is equal to

$$\frac{\pi^2}{6n^2 h_n^2} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right) \lambda_c(x + s) dx + o\left(\frac{1}{n^2 h_n}\right). \quad (3.11)$$

Since s is a Lebesgue point of λ_c (cf. (1.4)) and the kernel K has support in $[-1, 1]$, we see that the first term on the r.h.s. of (3.11) is equal to the r.h.s. of (2.9). We also see that the second term of (3.11) is of the same order as the second term on the r.h.s. of (2.9). This completes the proof of Theorem 4.

Proof of Theorem 1. By Lemma 1, Theorem 4 and the assumption $n^2 h_n \rightarrow \infty$ as $n \rightarrow \infty$, we have (2.6) and (2.7). This completes the proof of Theorem 1.

Proof of Theorem 2. To prove Theorem 2, we have to show, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}(|\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)| > \varepsilon) < \infty. \quad (3.12)$$

First note that

$$\begin{aligned} \mathbf{P}(|\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)| > \varepsilon) &\leq \mathbf{P}(|\hat{\lambda}_{c,n,K}(s) - \mathbf{E}\hat{\lambda}_{c,n,K}(s)| \\ &\quad + |\mathbf{E}\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)| > \varepsilon). \end{aligned}$$

By Lemma 1, there exists a positive real number M such that, for all $n > M$, $|\mathbf{E}\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)| < \varepsilon/2$. This implies, for sufficiently large n ,

$$\begin{aligned} \mathbf{P}(|\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)| > \varepsilon) &\leq \mathbf{P}\left(|\hat{\lambda}_{c,n,K}(s) - \mathbf{E}\hat{\lambda}_{c,n,K}(s)| > \frac{\varepsilon}{2}\right) \\ &\leq \frac{4\text{Var}(\hat{\lambda}_{c,n,K}(s))}{\varepsilon^2}, \end{aligned} \quad (3.13)$$

by the Chebyshev inequality. From Theorem 4 and the assumption $h_n = n^{-\alpha}$ with $0 < \alpha < 1$, we see that the r.h.s. of (3.13) is $\mathcal{O}(n^{\alpha-2})$, which is summable, and Theorem 2 follows. This completes the proof of Theorem 2.

References

- [1] R. Helmers and I W. Mangku, Estimating the intensity of a cyclic Poisson process in the presence of linear trend, *Ann. Inst. Statist. Math.* 61 (2009), 599-628.
- [2] R. Helmers, I W. Mangku and R. Zitikis, Statistical properties of a kernel-type estimator of the intensity function of a cyclic Poisson process, *J. Multivariate Anal.* 92 (2005), 1-23.
- [3] I W. Mangku, Consistent estimation of the distribution function and the density of waiting time of a cyclic Poisson process with linear trend, *Far East J. Theor. Stat.* 33 (2010), 81-91.
- [4] I W. Mangku, Estimating the intensity obtained as the product of a periodic function with the linear trend of a non-homogeneous Poisson process, *Far East J. Math. Sci. (FJMS)* 51 (2011), 141-150.
- [5] R. L. Wheeden and A. Zygmund, *Measure and Integral: An Introduction to Real Analysis*, Marcel Dekker, Inc., New York, 1977.

Editorial Board

Gunjan Agrawal, India	George S. Androulakis, Greece
Natig M. Atakishiyev, Mexico	Carlo Bardaro, Italy
Antonio Carbone, Italy	Yong Gao Chen, China
Hasan Coskun, USA	Claudio Cuevas, Brazil
Zhenlu Cui, USA	Maslina Darus, Malaysia
Manav Das, USA	Massimiliano Ferrara, Italy
Shusheng Fu, China	Salvatore Ganci, Italy
Wei Dong Gao, China	Demetris P. K. Ghikas, Greece
Toshio Horiuchi, Japan	Jay M. Jahangiri, USA
Lisa M. James, USA	Young Bae Jun, South Korea
Koji Kikuchi, Japan	Hideo Kojima, Japan
Victor N. Krivtsov, Russian Federation	Yangming Li, China
Alison Marr, USA	Manouchehr Misaghian, USA
Jong Seo Park, South Korea	Cheon Seoung Ryoo, South Korea
Alexandre J. Santana, Brazil	K. P. Shum, China
A. L. Smirnov, Russian Federation	Ashish K. Srivastava, USA
Chun-Lei Tang, China	E. Thandapani, India
Carl A. Toews, USA	B. C. Tripathy, India
Vladimir Tulovsky, USA	Mitsuru Uchiyama, Japan
Qing-Wen Wang, China	G. Brock Williams, USA
Peter Wong, USA	Chaohui Zhang, USA
Pu Zhang, China	Kewen Zhao, China

Principal Editor
Azad, K. K. (India)

CONTENTS

Chapter 1: Introduction to the Book (1-5)
Chapter 2: The History of the Subject (6-15)
Chapter 3: The Theory of the Subject (16-25)
Chapter 4: The Practice of the Subject (26-35)
Chapter 5: The Philosophy of the Subject (36-45)
Chapter 6: The Science of the Subject (46-55)
Chapter 7: The Art of the Subject (56-65)
Chapter 8: The Profession of the Subject (66-75)
Chapter 9: The Future of the Subject (76-85)
Chapter 10: The Conclusion (86-95)

INDEX

Index of Names (96-100)
Index of Subjects (101-110)
Index of Terms (111-120)
Index of Dates (121-130)
Index of Places (131-140)
Index of Things (141-150)