Pairing in Neutron Stars

Tony Sumaryada

(Under Supervision of Prof. A. Volya)

Department of Physics
Florida State University

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1 Introduction

Neutron star is an interesting object. It is dense with the mass of about the mass of the sun, while the radius of only about 10 km. This tremendous density may result in some unusual states of matter or lead to exotic many-body physics phenomena. For the discussion here we assume that a neutron star is a charge neutral system of nucleons and electrons in beta equilibrium at zero temperature. To sustain the stability, the neutron star must be in hydrodynamic equilibrium between gravity and Fermi degeneracy pressure. The mass, radius, pressure, and density profiles of neutron star are determined by general relativity and hydrodynamics of the neutron star matter consistently with its equation of state, via the so called Tolman-Oppenheimer-Volkov equation [1].

Given tremendous variations of density from almost zero on the surface to that of several times the nuclear density in the interior region the structure of neutron stars may be rich and diverse. In figure 1, we show the typical cross sections of the neutron star. Based on the increasing density of matter we roughly separate four different layers.

- The first layer is the outer crust; region A in Fig. 1 at low density it is made up from bare nuclei in thermodynamic equilibrium with electron bath. Although the structure here is expected to be a typical Fermi state exotic random or coherent lattice structures can not be excluded.

- We refer to the second layer as the inner crust, given higher density and shifted beta-equilibrium we expect it to be made up from more neutron rich nuclei that are embedded in the seas of neutrons and relativistic electrons. Region B in Fig. 1. The structure is expected to be seriously influenced by the neutron superconductivity involving both phenomena the internal nuclear superconductivity and neutron-pair transitions between nuclei. The nuclei themselves may be arranged in random configurations or even form regular lattice structures embedded in neutron superfluid and relativistic electron bath. Because of the competition between the coulomb and surface energies, nuclei in this region experience large shape fluctuations ranging from spherical shapes to rods, plates, or tubes [2].

- With the further increase in density and approaching the nuclear matter density we reach a more uniform neutron matter with only perhaps a view cavities we identify this as the outer core, regions C in Fig. 1. The superconducting state of neutron matter is expected to determine the equation of state here, however neutron bubbles especially if coherently arranged can be of importance [3].

- There is little known about the most inner core or the region D in our Fig. 1. There are speculations that higher densities will lead to exotic phases of pion, kaon, hyperon condensates and eventually to a quark matter. The transitions between these phases, relevant degrees of freedom and the role

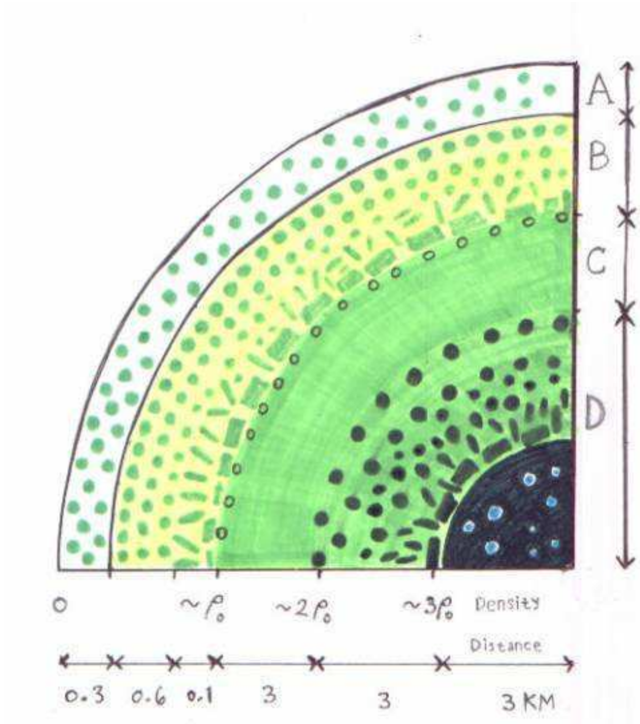

Figure 1: Structure of Neutron Star.

of many-body collective phenomena such as superconductivity for both structure and transitions between phases are open questions.

Our motivation in this work is to tackle some of the questions related to the structure of the neutron star. Clearly fully solving all of these problems is a task of enormous complexity and remains infeasible for the near future. In this work we will concentrate on the aspects related to pairing, regardless of degrees of freedom free nuclei, nucleons, mesons or quarks the two-body correlations of pairing type are extremely important. These correlations correspond to particle-particle scatterings in the lowest momentum channel or equivalently with the zero impact parameter. In our study we plan to traverse different regions of neutron star and discuss the role of superconducting correlations. For the crust region the questions of pairing in neutron rich nuclei, the pairing of nucleons in continuum of scattering states between nuclei, the role of pairing in forming the arrangement of nuclei such as random or ordered, and finally the question of equation of state of such dilute matter are of interest. The nuclear superconductivity in neutron rich nuclei and pairing in continuum unlike many other subjects related to neutron stars can be studied in the lab. For the more dense core regions the list of question includes the role of pairing for the general structure of matter that may involve the possibility of bubbles; the pairing correlations between different non-nucleonic degrees of freedom and the possibility of exotic pairing forms. Finally the role of pairing in possible existance and transition to the quark matter inside the neutron star is to be explored.

The structure of this presentation is as follows: In the following section, we show the importance of a short range interaction to produce a coherence effect using a simple analytically solvable model. In section 3, we concentrate on some pairing calculations for many-body system using BCS approach. We briefly outline the derivation of BCS equations and give some examples of the BCS calculation relevant to nuclear system. The future outlook and research plans are presented in section 4.

2 Coherence effect due to the short range interaction

To show the effect of short range interaction on a system we consider a two-body problem with a relative coordinate x. The two particles under consideration interact and form a confined system, this interaction is given by the Hamiltonian H_0, and the discreet set of states of this system is given by energies ε_n and wave functions $\phi_n(x)$

$$H_0\phi_n = \varepsilon_n\phi_n. \tag{1}$$

We suppose that in addition to this there is a short range interaction, given here by the Dirac delta function with some strength g. The role of this perturbation is the question to be addressed. The full Hamiltonian for the system is

$$H = H_0 - g\delta(x). \tag{2}$$

In order to solve the full Shrödinger equation

$$H\psi(x) = E\psi(x) \tag{3}$$

we decompose the eigenstate $\psi(x)$ in the unperturbed basis

$$\psi(x) = \sum_{n=0}^{\infty} c_n\phi_n(x) \tag{4}$$

After substitution of Eq. (4) into Eq. (2) the eigenvalue matrix diagonalization equation becomes :

$$c_k\varepsilon_k - \sum_{k,n=0}^{\infty} g\,\phi_k(0)\phi_n(0)c_n = Ec_k. \tag{5}$$

The Hamiltonian matrix here is separable and therefore allows for an analytic solution. Introducing the collective field Δ

$$\Delta = \sum_{n=0}^{\infty} \phi_n(0)c_n \tag{6}$$

we obtain expansion coefficient c_k as

$$c_k = \frac{g\phi_k(0)\Delta}{\varepsilon_k - E} \tag{7}$$

then self-consistency of this equation and Eq. (6) result is a secular equation for the energy

$$\frac{1}{g} = \sum_{k=0}^{\infty} \frac{\phi_k^2(0)}{\varepsilon_k - E} \tag{8}$$

This equation is known to represent a rather generic phenomenon describing the collectivity formation. When $g = 0$ the roots coincide with unperturbed roots ε_k. For non zero g with a graphical solution one can establish that each root E_k lies between ε_{k-1} and ε_k, we assume here attractive $g > 0$ interaction. The ground state E_0, however, is not bound from below and typically due to coherent level repulsion from all other states is significantly pushed down. If all states are degenerate $\varepsilon_k = 0$ and with approximation that $\phi_k^2(0) \equiv \phi^2$ are independent of k the ground state of perturbed system is found at $E_0 = -g\phi^2 N$. The repulsion is coherent since it is proportional to the macroscopically large number of states in the system N.

An example of Harmonic oscillator below illustrates the situation. Consider only even k due to the parity, the sum in Eq. (8) can be done analytically which leads to

$$\frac{g}{2}\,\frac{\Gamma(\frac{-E}{2})}{\Gamma(\frac{1-E}{2})} = 1 \tag{9}$$

In Fig.2(a), we show the energy spectrum as a function of pairing constant g. Before pairing ($g = 0$) we simply get the spectrum of energy of harmonic oscillator $E_k \to \varepsilon_k = k + 1/2$. When we turn on the pairing constant g, the ground state goes down in energy $E_0 \to -\infty$ and forms a single coherent state (pairing state), while the other states asymptotically go down to the level below $E_k \to \varepsilon_{k-1}$. The wave functions corresponding to the ground states of perturbed and unperturbed systems are shown in Fig.2(b). It is clear that the short range correlation forces particle confinement leading to a sharp peak near $x = 0$ as expected from the delta potential.

3 Superconductivity, and the BCS Theory

For the case of the many-body system with short range pairwise interaction the macroscopic theory was developed by Bardeen, Cooper, and Schrieffer in 1957, known as BCS theory [4]. The BCS theory represents a variational approach where many-body wave function is assumed to be a product of particle-pairs. Pairwise structure is favored by the attractive short range interaction, the effect considered in the previous section.

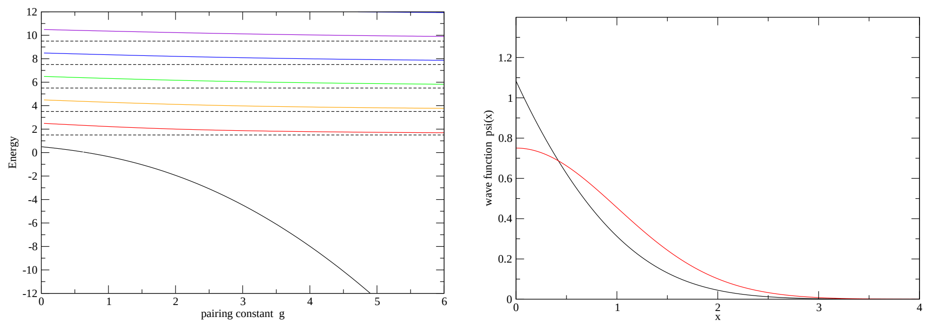

Figure 2: (a)Energy spectrum of harmonic oscillator as a function of pairing constant g and (b) the ground state wave function of harmonic oscillator before and after pairing with g=1.00

3.1 BCS theory formulation

In this section we outline the BCS approach, the more detailed consideration can be found in various textbook [5].

The concept of a pair is central for the BCS construction. Physics of interaction dictates what is to be taken as a pair, namely in what channel the attractive interaction between particle is the strongest. In this analysis we consider the most common and the simplest case when pair represents two particles on time conjugate states $(k, \tilde{k})$. The k here labels single particle quantum state. It is natural that short range interaction would be strongest for these angular momentum $L = 0$ head on collisions. The exotic pairs with other degrees of freedom such as isospin or color are to be addressed in future work. The scattering in this pairing channel is described by the matrix element $V_{kk'}$ reflecting transition of initial state $(k',\tilde{k'})$ to the final state $(k, \tilde{k})$. It is useful to use second quantization language, where we introduce creation and annihilation operators $a_k^\dagger$ and a_k; the typical fermion commutation rules are

$$\{a_k, a_{k'}^\dagger\} = a_k a_{k'}^\dagger + a_{k'}^\dagger a_k = \delta_{kk'}, \quad \{a_k, a_{k'}\} = a_k a_{k'} + a_{k'} a_k = 0. \tag{10}$$

we also introduce the operators:

$$p_k = a_{\tilde{k}} a_k, \quad p_k^\dagger = a_k^\dagger a_{\tilde{k}}^\dagger, \qquad n_k = a_k^\dagger a_k, \quad n_{\tilde{k}} = a_{\tilde{k}}^\dagger a_{\tilde{k}} \tag{11}$$

which satisfy the commutation rules below:

$$[p_k, p_{k'}^\dagger] = \delta_{kk'}(1 - n_{k'} - n_{\tilde{k'}}) \tag{12}$$

It should be noted that, trough out this paper we are going to use the same notations for both the operators and their expectation values.

Now, assume that the Hamiltonian for the system is

$$H = \sum_k \varepsilon_k n_k + \sum_{kk'>0} V_{kk'} p_k^\dagger p_{k'} \tag{13}$$

which includes no interactions other than pairing. To get the ground state we consider wave functions

$$|BCS\rangle = \prod_{k>0}^{\infty}(u_k + v_k p_k^{\dagger})|0\rangle \tag{14}$$

with variational parameters u_k and v_k. Imposing the normalization on BCS wave function, we get

$$u_k^2 + v_k^2 = 1. \tag{15}$$

We can interpret the v_k^2 as the probability of pairing state $(k, \tilde{k})$ being occupied, while u_k^2 is the probability of that state to remain empty. We also define the particle number operator N as

$$N = \sum_{k>0}(n_k + n_{\tilde{k}}) \tag{16}$$

The BCS wave function (14) explicitly doesn't conserve the particle number, however in the macroscopic case asymptotically exact restoration of particle number conservation can be reached by introducing chemical potential λ adjusted so that $\langle BCS|\hat{N}|BCS\rangle = N$. The Hamiltonian then changed as $H \to H - \lambda\hat{N}$. By using $n_k = v_k^2$ as a free parameter, we minimize energy

$$E = \langle BCS|H|BCS\rangle = 2\sum_{k>0}\varepsilon_k^0 v_k^2 + \sum_{kk'>0} V_{kk'}u_k v_k u_{k'} v_{k'} + \sum_{k>0} V_{kk} v_k^2 \tag{17}$$

which leads to

$$\varepsilon_k + \frac{(1-2n_k)}{2\sqrt{(1-n_k)n_k}}\sum_{k'>0} V_{kk'}u_{k'}v_{k'} = 0 \tag{18}$$

Similar to the discussion conducted in the previous section this set of equations can be solved with the introduction of collective variable called gap parameter

$$\Delta_k = \sum_{k'>0} V_{kk'}u_{k'}v_{k'} \tag{19}$$

then we have

$$\varepsilon_k + \frac{(1-2n_k)}{2\sqrt{(1-n_k)n_k}}\Delta_k = 0, \tag{20}$$

which solved for n_k gives

$$n_k = \frac{1}{2}(1 - \frac{\varepsilon_k}{e_k}). \tag{21}$$

where the quasiparticle energy is introduced as

$$e_k = \sqrt{\varepsilon_k^2 + \Delta_k^2} \tag{22}$$

Thus

$$v_k^2 = \frac{1}{2}(1 - \frac{\varepsilon_k}{e_k}), \quad u_k^2 = \frac{1}{2}(1 + \frac{\varepsilon_k}{e_k}). \tag{23}$$

In full similarity to the two-body problem of the previous section the self-consistency in collective gap Δ through equations (19) , (23) , leads to

$$\Delta_k = \frac{1}{2}\sum_{k'>0}\frac{\Delta_{k'}}{e_{k'}}V_{kk'}. \tag{24}$$

This result is called gap equation. The Eq. (24) is simplified if $V_{kk'}$ is constant, in this case the gap Δ is state independent. In the following section we consider some BCS applications.

3.2 Examples of BCS calculations

3.2.1 One level problem/degenerate states

One of the simplest pairing problems is the case of N particles in the degenerate valence space so that all states k have the same energy $\varepsilon_k = \varepsilon$. In realistic situation this can occur due to symmetries such as the case of a single particle level with large spin j that has $2j+1$ degenerate states labeled by magnetic quantum number. The interaction in this case is also constant $V_{kk'} = V$. The BCS approach here leads to two equations: gap equation and equation for the chemical potential that sets the particle number.

$$\Delta = \frac{\Omega V}{2}\left(\frac{\Delta}{\sqrt{(\varepsilon-\lambda)^2+\Delta^2}}\right), \quad N = \frac{\Omega}{2}\left(1-\frac{(\varepsilon-\lambda)}{\sqrt{(\varepsilon-\lambda)^2+\Delta^2}}\right). \tag{25}$$

The solution of this set is

$$\Delta = \frac{V}{2}\sqrt{N\Omega - N^2}, \quad \lambda = \varepsilon - V\left(\frac{\Omega}{2} - N\right). \tag{26}$$

For the strong pairing where one can neglect the differences between single particle energies the above equations reflect a typical parabolic behavior of the gap and linear increase in chemical potential as the valence space is gradually filled. The BCS total energy for the degenerate model is:

$$E_{BCS}(N,s) = N\varepsilon + \frac{V}{4}\left[(N-s)(\Omega - N - s + 2\frac{(N-s)}{(\Omega-2s)})\right] \tag{27}$$

with s is the number of unpaired particles.
We can also solve the pairing problem in degenerate model by using exact pairing method. Introducing operator P, $P^\dagger$, and N :

$$P = \sum_k p_k \ , \qquad P^\dagger = \sum_k p_k^\dagger \ , \qquad N = \sum_k n_k \tag{28}$$

which have the commutation relations :

$$[P^\dagger, P] = N - \frac{\Omega}{2} \ , \qquad [N, P] = -2P \ , \qquad [N, P^\dagger] = 2P^\dagger \tag{29}$$

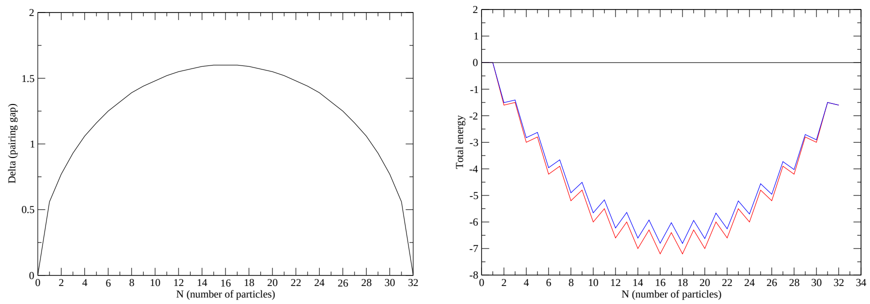

Figure 3: (a) Pairing gap vs N (b) total energies vs N, BCS result in blue and exact result in red color

the Hamiltonian of equation (13), now can be written in more simple way as :

$$H = N\varepsilon + VP^{\dagger}P \tag{30}$$

To get the total energy, we can map the problem into the problem of particles in the field by analoging the operator above with the angular momentum operators

$$P \;\rightarrow\; L_- \;, \qquad P^{\dagger} \;\rightarrow\; L_+ \;, \qquad \frac{1}{2}(N - \frac{\Omega}{2}) \rightarrow\; L_z \tag{31}$$

and finally the total energy of exact pairing method is:

$$E_{exact}(N, s) = N\varepsilon + \frac{V}{4}\left[(N - s)(\Omega - N - s + 2)\right] \tag{32}$$

We can relate the expectation value of angular momentum with the number of unpaired particles s as :

$$s = 2(\frac{\Omega}{4} - L) \tag{33}$$

In Fig.3(a), the gap is plotted as a function of particle number for the degenerate model. In order to give the problem some realistic flavor we select the valence space $\Omega = 32$ to represent a set of states $h_{11/2}$, $d_{3/2}$, $s_{1/2}$, $g_{7/2}$, and $d_{5/2}$ which are close in energy and can roughly be considered as degenerate. Within this shell we study the problem of neutron pairing for ^{100}Sn- ^{132}Sn. The typical pairing strength is V=-0.1MeV [6]. In Fig. 3(b), we compare the BCS total energy (the blue line) with the exact pairing energy (the red line) as a function of the number particles for the case $\varepsilon = 0, N = \Omega = 32$, and V= -0.1 MeV. We see clearly that the exact method always give you lower energy compared to BCS/variational calculation.

3.2.2 Two levels problem

The degenerate model discussed above represents the case of strong pairing and thus is somewhat limited as pair excitation to unoccupied orbitals are not

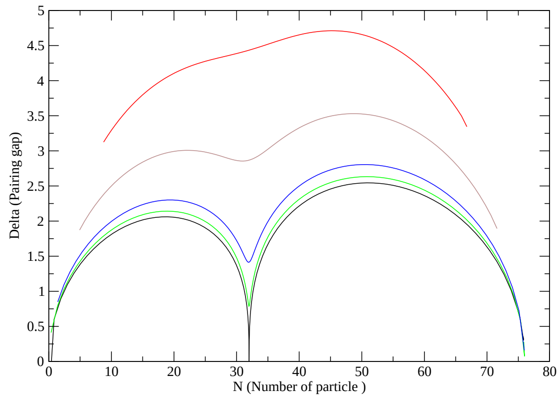

Figure 4: Pairing gap vs the number of particles for two levels problem

penalized by the requirement to overcome single particle energy differences. For this reason degenerate model has no pairing phase transition, even at weak pairing paired state is always energetically favored. The most general picture can be obtained in the two-level case. Here we consider two shells with energies ε_0 and ε_1. the degeneracies are Ω_0 and Ω_1, respectively [7] Without loosing any features we can assume constant V for all pair interactions within the shells and between the shells. As indicated earlier this assumptions leads to a single gap parameter.

For the example of the calculation shown in Fig.4, we again attempt to describe the realistic picture of two shells, that would correspond to tin isotopes in the region ^{100}Sn- ^{174}Sn. For the single particle energies we assume constant potential V which come from Woods-Saxon solution of the nuclear mean field potential adjusted for this region. We select various interactions strengths V = 92.4, 95, 100, 120, and 150 KeV. By increasing the pairing potential, we simply transform system from two separate shells to a single big shell.

4 Research Plans and outlook

In this presentation we report our first step toward understanding the role of coherent pairing effects on the structure and phases of matter in the neutron star. The importance of short range correlations and their role in forming the bound Cooper pairs has been shown in Sec. 2. This analytically solvable example is of particular importance as real coordinate space interaction was used. Obtaining the interaction matrix elements for the relevant pairing mode from fundamental laws will be crucial part for this project. In section 3, we have shown the study of the role of pairing in many-body systems. The basic formalism is outlined in Sec 3.1 , however future applications most likely will require its extension to include various phases of pairing coherence such as different color modes or isospin modes of pairing. Examples in Sec 3.2 show the needs for the development of fast and reliable general code to tackle pairing problem in a larger system. In terms of applications we plan to start with the study of pairing in typical nuclear systems, given a lot of experimental data and large amount of research done in this direction. This study will extend into the realm of neutron rich nuclei; here the interplay of pairing effects and neutron decay is the key question. By considering a collection of neutron rich nuclei and assuming their random, regular, or special distribution, we hope to model the crust of the neutron star and its properties. The variational procedure here would extent beyond BCS to include the properties of the distribution of nuclei, hopefully this can shed some light on the possibility of exotic structures such as planes, rods and etc. This work with nucleon degrees of freedom can be further extended into the outer core regions of neutron stars where neutron matter equation of state can be investigated along with the possibility and/or distribution of bubbles in it at lower density.

The meson and quark degrees of freedom are the next challenge in further steps toward understanding the physics of the neutron star's core. We will start by focusing on a "multi-flavor pairing" where Cooper pairs of different symmetries are possible. The consideration of analogous problems of SU(2)xSU(2) pairing in nuclear systems with rotational invariance and isospin is particularly valuable and practically important since interplay between isovector and isoscalar pairing modes in nuclei is still not fully resolved. Using the guiding principles of these cases as well as the examples from the condensed matter physics we plan to explore the superconductivity in the quark matter. By combining the quark deconfinement process with pairing effects on both hadron and quark levels we hope to get some idea about the phases and equation of state of the neutron star's core.

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