APPLICATIONS OF FLUID DYNAMICS BASED ON GAUGE FIELD THEORY APPROACH

A thesis submitted to the Fakultas Pasca-Sarjana Universitas Indonesia in partial fulfillment of the requirements for the degree of Master of Science
Graduate Program in Pure and Applied Physics

KETUT EKO ARI SAPUTRO
6302020149

Universitas Indonesia

2005

Lembar Persetujuan

Judul tesis	:	Applications Of Fluid Dynamics Based On Gauge Field Theory Approach
Nama	:	Ketut Eko Ari Saputro
NPM	:	6302020149

tesis ini telah diperiksa dan disetujui

Depok, 29 Juni 2005
Mengesahkan

Pembimbing I	Pembimbing II
(Dr. Terry Mart)	(Dr. L.T. Handoko)

Penguji I	Penguji II	Penguji III
(Dr. Agus Salam)	(Dr. Imam)	(Dr. Anto Sulaksono)

Ketua Program Magister Fisika
Program Pasca Sarjana FMIPA UI

(Dr. Dedi Suyanto)

Acknowledgements

I began working on my thesis soon as I joined the nonlinear physics group at University of Indonesia on July 2004 under supervision of Dr. L.T. Handoko. He was actually one of the reviewers for my BSc theses at Bogor Institute of Agricultural about the phenomenon on nonlinear optics. After that, I spent my time to study gauge field theory (it's something new for me).

First of all, I would like to acknowledge my primary advisor, Dr. L.T. Handoko for all guidance, patience and critical comments. I am also indebted to Dr Terry Mart, he taught me about the quantum mechanics. I thank D r.Anto Sulaksono and Dr. Agus Salam for his intriguing question during my defense. I would like to appreciate our theoretical group, Albert Sulaiman, Fahd, Jani, Ardi Mustafa, Anton, Freddy, etc for so many valuable discussion.

The last but not the least, Tri Apriyani, Dhifa al-hakim and Jauza Nur Azizah for their patience and I always love you all.
My study was supported by Nurul Fikri learning center.

Jakarta, 2005
Ketut eko Ari Saputro

Abstract

Recently, a new approach to deal with the Navier-Stokes equation has been developed. The equation which governs the fluid dynamics is a a non-linear one and then generally unsolvable. In the new approach, the fluid dynamics is described using the relativistic gauge invariant bosonic lagrangian which could reproduce the Navier-Stokes equation as its equation of motion through the Euler-Lagrange principle. Based on the lagrangian we model the fluid dynamics phenomenon as scattering of either three or four fluid bunches represented by the gauge field $A_\mu = (\phi, \vec{A}) \equiv \left(\frac{d}{2}\left|\vec{v}\right|^2 - V, -d\vec{v}\right)$ with $\vec{v}$ is velocity and d is a parameter to adjust the dimension for any potential V. Further we present all relevant Fynman rules and diagrams, and also provide complete calculations for all vertices induced by three and four fluid fields interactions.

References: 38 (1985-2003)

Contents

Chapter 1

Introduction

1.1 Background

The understanding of fluid dynamics like hydrodynamic turbulence is an important problem for nature science, from both, theoretical and experimental point of view, and has been investigated intensively over the last century. However a deep and fully comprehension of the problem remains obscure. Over the last years, the investigation of turbulent hydrodynamics has experienced a revival since turbulence has became a very fruitful research field for theorists who study the analogies between turbulence and field theory, critical phenomena and condensed matter physics , renewing the optimism to solve the turbulence problem. The dynamics of turbulent viscous fluid is expressed by the Navier-Stokes (NS) equations of motion, which in a vectorial form is a fluid flow described by,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \nu\vec{\nabla}^2\vec{v} \ , \tag{1.1}$$

where $\vec{v}$ is the velocity field, P is pressure, ρ is density and ν is the kinematic viscosity. The equation of continuity reduces to the requirement that the velocity field is divergenceless for incompressible fluids,

$$\vec{\nabla} \cdot \vec{v} = 0 \ . \tag{1.2}$$

In this context,

the hydrodynamic turbulence has attracted an enormous interest due to the universal characteristics stressed by an incompressible fluid with high Reynolds numbers in the fully developed turbulent regime. The Reynolds number, $R = L/U$. (where L is the integral length-scale of the largest eddies and U is a characteristic large-scale velocity), measures the competition between convective and diffusive processes in an incompressible fluid described by the NS equations. In view of this, the incompressible fluid flow assumes high Reynolds numbers when the velocity increases and, consequently, the solution for Eq. (1.1) becomes unstable and the fluid switches to a new regime of a very complex motion with the velocity varying almost randomly and without any noticeable order. To discover the laws describing what exactly is going on with the fluid in this turbulent regime is very important to both theoretical and applied science. Recently, A. Sulaiman and L.T. Handoko proposed an alternative approach to treat fluid dynamics [1]. In the fluid dynamics which is governed by the NS equation we are mostly interested only in how the forces are mediated, and not in the transition of an initial state to another final state as concerned in particle physics. Based on lagrangian density Navier-stokes gauge field theory, they describes the dynamical of interactions multi fluids system like on particle physics.

$$\mathcal{L}_{\rm NS} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - g J^{a\mu} A^a_\mu \ . \qquad (1.3)$$

In this thesis , we make a further investigation of the physical contents present in that theory, in order to furnish a better understanding of fluid dynamics.

1.2 Overview

This theses is organized as follow. The introduction and background of the problem are given in chapter one. Then the theoretical basic of this thesis, *i.e.*constructing the Navier-stokes equation from gauge field theory, will be described in chapter two. The main part is presented in chapter three. The discussion will be given in chapter four. The last chapter is devoted for summary.

Chapter 2

Navier-Stokes Equation From Gauge field theory

In this chapter we will construct the Navier-Stokes equation from first principle using relativistic bosonic lagrangian which is invariant under local gauge transformations. We show that by defining the bosonic field to represent the dynamic of fluid in a particular form, a general Navier-Stokes equation with conservative forces can be reproduced exactly. It also induces two new forces, one is relevant for rotational fluid, and the other is due to the fluid's current or density. This approach provides an underlying theory to apply the tools in field theory to the problems in fluid dynamics.

2.1 Introduction

The Navier-Stokes (NS) equation represents a non-linier system with flow's velocity $\vec{v} \equiv \vec{v}(x_\mu)$, where x_μ is a 4-dimensional space consists of time and spatial spaces, $x_\mu \equiv (x_0, x_i) = (t, \vec{r}) = (t, x, y, z)$. Note that throughout the paper we use natural unit, *i.e.* the light velocity $c = 1$ such that $ct = t$ and then time and spatial spaces have a same dimension. Also we use the relativistic (Minkowski) space, with the metric $g_{\mu\nu} = (1, -\vec{1}) = (1, -1, -1, -1)$ that leads to $x^2 = x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu = x_0^2 - \mathbf{x}^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$.

Since the NS equation is derived from the second Newton's law, in principle it should be derived from analytical mechanics using the principle of at-least action on the hamiltonian as already done in several papers [2]. Some papers also relate it with the Maxwell equation [3]. The relation between NS and Maxwell equations is, however, not clear and intuitively understandable claim since both equations represent different systems. Moreover, some authors have also formulated the fluid dynamics in lagrangian with gauge symmetries [4]. However, in those previous works the lagrangian has been constructed from continuity equation.

Inspired by those pioneering works, we have tried to construct the NS equation from first principle of analytical mechanics, *i.e.* starting from lagrangian density. Also concerning that the NS equation is a system with 4-dimensional space as mentioned above, it is natural to borrow the methods in the relativistic field theory which treats time and space equally. Then we start with developing a lagrangian for bosonic field and put a contraint such that it is gauge invariant. Taking the bosonic field to have a particular form representing the dynamics of fluid, we derive the equation of motion which reproduces the NS equation.

2.2 Gauge invariant bosonic lagrangian

In the relativistic field theory, the lagrangian (density) for a bosonic field A is written as [6],

$$\mathcal{L}_A = (\partial^\mu A)(\partial_\mu A) + m_A^2 A^2 \ , \tag{2.1}$$

where m_A is a coupling constant with mass dimension, and $\partial_\mu \equiv \partial/\partial x^\mu$. The bosonic field has the dimension of $[A] = 1$ in the unit of mass dimension $[m] = 1$ ($[x_\mu] = -1$). The bosonic particles are, in particle physics, interpreted as the particles which are responsible to mediate the forces between interacting fermions, ψ's. Then, one has to first start from the fermionic lagrangian,

$$\mathcal{L}_\psi = i\overline{\psi}\gamma^\mu(\partial_\mu \psi) - m_\psi \overline{\psi}\psi \ , \tag{2.2}$$

where ψ and $\overline{\psi}$ are the fermion and anti-fermion fields with the dimension $[\psi] = [\overline{\psi}] = 3/2$ (then $[m_\psi] = 1$ as above), while γ^μ is the Dirac gamma matrices. In order to expand the theory and incorporate some particular interactions, one should impose some symmetries.

2.2.1 Abelian gauge theory

For simplicity, one might introduce the simplest symmetry called $U(1)$ (abelian) gauge symmetry. The $U(1)$ local transformation[1] is just a phase transformation $U \equiv \exp[-i\theta(x)]$ of the fermions, that is $\psi \xrightarrow{U} \psi' \equiv U\,\psi$. If one requires that the lagrangian in Eq. (2.2) is invariant under this local transformation, *i.e.* $\mathcal{L} \to \mathcal{L}' = \mathcal{L}$, a new term coming from replacing the partial derivative with the covariant one $\partial_\mu \to D_\mu \equiv \partial_\mu + ieA_\mu$, should be added as,

$$\mathcal{L} = \mathcal{L}_\psi - e(\overline{\psi}\gamma^\mu\psi)A_\mu \ . \tag{2.3}$$

Here the additional field A_μ should be a vector boson since $[A_\mu] = 1$ as shown in Eq. (2.1). This field is known as gauge boson and should be transformed under $U(1)$ as,

$$A_\mu \xrightarrow{U} A'_\mu \equiv A_\mu + \frac{1}{e}(\partial_\mu\theta) \ , \tag{2.4}$$

to keep the invariance of Eq. (2.3). Here e is a dimensionless coupling constant interpreted as electric charge later on.

The existence of a particle requires that there must be a kinetic term of that particle in the lagrangian. In the case of newly introduced A_μ above, it is fulfilled by adding the kinetic term using the standard boson lagrangian in Eq. (2.1). However, it is easy to verify that the kinetic term (*i.e.* the first term) in Eq. (2.1) is not invariant under the transformation of Eq. (2.4). Then one must modify the kinetic term to keep the gauge invariance. This can be done by writing down the kinetic term in the form of anti-symmetric strength tensor $F_{\mu\nu}$ [5],

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{2.5}$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, and the factor of 1/4 is just a normalization factor.

On the other hand, the mass term (the second term) in Eq. (2.1) is automatically discarded in this theory since the quadratic term of A_μ is not invariant (and then not allowed) under transformation in Eq. (2.4). In particle physics this result justifies the interpretation of gauge boson A_μ as photon which is a massless particle.

[1] The terminology "local" here means that the parameter θ is space dependent, *i.e.* $\theta \equiv \theta(x)$. One needs also to put a preassumption that the transformation is infinitesimal, *i.e.* $\theta \ll 1$.

Finally, imposing the $U(1)$ gauge symmetry, one ends up with the relativistic version of electromagnetic theory, known as the quantum electrodynamics (QED),

$$\mathcal{L}_{\text{QED}} = i\overline{\psi}\gamma^\mu(\partial_\mu\psi) - m_\psi\overline{\psi}\psi - eJ^\mu A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{2.6}$$

where $J^\mu \equiv \overline{\psi}\gamma^\mu\psi = (\rho, \vec{J}) = (J_0, \vec{J})$ is the 4-vector current of fermion which satisfies the continuity equation, $\partial_\mu J^\mu = 0$, using the Dirac equation governs the fermionic field [6].

2.2.2 Non-abelian gauge theory

One can furthermore generalize this method by introducing a larger symmetry. This (so-called) non-abelian transformation can be written as $U \equiv \exp\left[-iT^a\theta^a(x)\right]$, where T^a's are matrices called generators belong to a particular Lie group and satisfy certain commutation relation like $[T^a, T^b] = if^{abc}T^c$, where the anti-symmetric constant f^{abc} is called the structure function of the group [7]. For an example, a special-unitary Lie group $SU(n)$ has $n^2 - 1$ generators, and the subscripts a, b, c run over $1, \cdots, n^2 - 1$.

Following exactly the same procedure as Sec. 2.2.1, one can construct an invariant lagrangian under this transformation. The differences come only from the non-commutativeness of the generators. This induces $\partial_\mu \to D_\mu \equiv \partial_\mu + igT^aA^a_\mu$, and the non-zero f^{abc} modifies Eq. (2.4) and the strength tensor $F_{\mu\nu}$ to,

$$A^a_\mu \xrightarrow{U} A^{a\prime}_\mu \equiv A^a_\mu + \frac{1}{g}(\partial_\mu\theta^a) + f^{abc}\theta^b A^c_\mu \ , \tag{2.7}$$

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu \ , \tag{2.8}$$

where g is a particular coupling constant as before. One then has the non-abelian (NA) gauge invariant lagrangian that is analoguous to Eq. (2.6),

$$\mathcal{L}_{\text{NA}} = i\overline{\psi}\gamma^\mu(\partial_\mu\psi) - m_\psi\overline{\psi}\psi - gJ^{a\mu}A^a_\mu - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \ , \tag{2.9}$$

while $J^{a\mu} \equiv \overline{\psi}\gamma^\mu T^a\psi$, and this again satisfies the continuity equation $\partial_\mu J^{a\mu} = 0$ as before. For instance, in the case of $SU(3)$ one knows the quantum chromodynamics (QCD) to explain the strong interaction by introducing eight gauge bosons called gluons induced by its eight generators.

2.3 The NS equation from the gauge field theory

In the fluid dynamics which is governed by the NS equation we are mostly interested only in how the forces are mediated, and not in the transition of an initial state to another final state as concerned in particle physics. Within this interest, we need to consider only the bosonic terms in the total lagrangian. Assuming that the lagrangian is invariant under certain gauge symmetry explained in the preceding section, we have,

$$\mathcal{L}_{\rm NS} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - g J^{a\mu} A^a_\mu \; . \tag{2.10}$$

We put an attention on the current in second term. It should not be considered as the fermionic current as its original version, since we do not introduce any fermion in our system. For time being we must consider $J^{a\mu}$ as just a 4-vector current, and it is induced by different mechanism than the internal interaction in the fluid represented by field A^a_μ. Actually it is not a big deal to even put $J^{a\mu} = 0$ (free field lagrangian), or any arbitrary forms as long as the continuity equation $\partial_\mu J^{a\mu} = 0$ is kept.

According to the principle of at-least action for the action $S = \int {\rm d}^4 x \, \mathcal{L}_{\rm NS}$, *i.e.* $\delta S = 0$, one obtains the Euler-Lagrange equation,

$$\partial_\mu \frac{\partial \mathcal{L}_{\rm NS}}{\partial(\partial^\mu A^a_\nu)} - \frac{\partial \mathcal{L}_{\rm NS}}{\partial A^a_\nu} = 0 \; . \tag{2.11}$$

Substituting Eq. (2.10) into Eq. (2.11), this leads to the equation of motion (EOM) in term of field A^a_μ,

$$\partial_\mu (\partial^\nu A^a_\nu) - \partial^2 A^a_\mu + g J^{a\mu} = 0 \; . \tag{2.12}$$

If A_μ is considered as a field representing a fluid system for each a, then we have multi fluids system governed by a single form of EOM. Inversely, the current can be derived from Eq. (2.12) to get,

$$J^{a\mu} = -\frac{1}{g} \partial^\nu \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu \right) \; , \tag{2.13}$$

and one can easily verify that the continuity equation is kept. We note that this equation holds for both abelian and non-abelian cases, since the last term in Eq. (2.8) contributes nothing due to its anti-symmetry. Also, this reproduces the relativistic version of the classical electromagnetic density and current of Maxwell.

The next task is to rewrite the above EOM to the familiar NS equation. Let us first consider a single field A_μ. Then the task can be accomplished by defining the field A_μ in term of scalar and vector potentials,

$$\begin{aligned} A_\mu &= (A_0, A_i) = (\phi, \vec{A}) \\ &\equiv \left(\frac{d}{2} \left| \vec{v} \right|^2 - V, -d\vec{v} \right) , \end{aligned} \tag{2.14}$$

where d is an arbitrary parameter with the dimension $[d] = 1$ to keep correct dimension for each element of A_μ. $V = V(\vec{r})$ is any potential induced by conservative forces. The condition for a conservative force $\vec{F}$ is $\oint \mathrm{d}\vec{r} \cdot \vec{F} = 0$ with the solution $\vec{F} = \vec{\nabla}\phi$. This means that the potential V must not contain a derivative of spatial space. We are now going to prove that this choice is correct.

From Eq. (2.12) it is straightforward to obtain,

$$\partial_\mu A^a_\nu - \partial_\nu A^a_\mu = -g \oint \mathrm{d}x_\nu J^{a\mu} . \tag{2.15}$$

First we can perform the calculation for $\mu = \nu$ where we obtain trivial relation, that is $J^{a\mu} = 0$. Non-trivial relation is obtained for $\mu \neq \nu$,

$$\partial_0 A_i - \partial_i A_0 = g \oint \mathrm{d}x_0 J_i = -g \oint \mathrm{d}x_i J_0 . \tag{2.16}$$

Different sign in the right hand side merely reflects the Minkowski metric we use. Now we are ready to derive the NS equation. Substituting the 4-vector potensial in Eq. (2.14) into Eq. (2.16), we obtain $d\,\partial_0 v_i + \partial_i \phi = g\tilde{J}_i$ or,

$$d\,\partial_0 \vec{v} + \vec{\nabla}\phi = g\vec{\tilde{J}} , \tag{2.17}$$

where $\tilde{J}_i \equiv -\oint \mathrm{d}x_0 J_i = \oint \mathrm{d}x_i J_0$. Using the scalar potential given in Eq. (2.14), we obtain,

$$d\,\frac{\partial \vec{v}}{\partial t} + \frac{d}{2}\vec{\nabla} \left| \vec{v} \right|^2 - \vec{\nabla}V = g\vec{\tilde{J}} . \tag{2.18}$$

By utilizing the identity $\frac{1}{2}\vec{\nabla} \left| \vec{v} \right|^2 = (\vec{v} \cdot \vec{\nabla})\vec{v} + \vec{v} \times (\vec{\nabla} \times \vec{v})$, we arrive at,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \frac{1}{d}\vec{\nabla}V - \vec{v} \times \vec{\omega} + \frac{g}{d}\vec{\tilde{J}} , \tag{2.19}$$

where $\vec{\omega} \equiv \vec{\nabla} \times \vec{v}$ is the vorticity. This result reproduces a general NS equation with arbitrary conservative forces ($\vec{\nabla}V$) and some additional forces. This result justifies our choice for the bosonic field in Eq. (2.14).

Just to mentioned, the potential could represent the already known ones such as,

$$V(\vec{r}) = \begin{cases} P(\vec{r})/\rho(\vec{r}) & : \quad \text{pressure} \\ Gm/|\vec{r}| & : \quad \text{gravitation} \\ (\nu+\eta)(\vec{\nabla}\cdot\vec{v}) & : \quad \text{viscosity} \end{cases} . \tag{2.20}$$

Here, $P, \rho, G, \nu + \eta$ denote pressure, density, gravitational constant and viscosity as well. We are able to extract a general force of viscosity, $\vec{\nabla} V_{\text{viscosity}} = \eta \vec{\nabla} \left(\vec{\nabla} \cdot \vec{v} \right) + \nu \left(\vec{\nabla}^2 \vec{v} \right) + \nu \left(\vec{\nabla} \times \vec{\omega} \right)$ using the identity $\vec{\nabla}(\vec{\nabla} \cdot \vec{v}) = \vec{\nabla} \times \vec{\omega} + \vec{\nabla}^2 \vec{v}$. This reproduces two terms relevant for both compressible and incompressible fluids, while the last term contributes to the rotational fluid for non-zero $\vec{\omega}$. This provides a natural reason for causality relation between viscosity and turbulence as stated in the definition of Reynold number, $R \propto \nu^{-1}$.

A general NS equation of multi fluids system can finally be obtained by putting the superscript a back to the equation,

$$\frac{\partial \vec{v}^a}{\partial t} + (\vec{v}^a \cdot \vec{\nabla})\vec{v}^a = \frac{1}{d}\vec{\nabla} V^a - \vec{v}^a \times \vec{\omega}^a + \frac{g}{d}\vec{J}^a , \tag{2.21}$$

Here the second term in the right hand side is a new force relevant for rotational fluid, while the last term is due to the current or density of fluid.

We would like to note an important issue here. One can take arbitrary current forces in the NS equation (Eq. (2.21)), as long as the continuity equation is kept, but should set a small number for g. This is very crucial since we will use the perturbation method of field theory to perform any calculation in fluid dynamics starting from the lagrangian in Eq. (2.10) later on. Taking arbitrary and small enough coupling constant ($g \ll 1$) is needed to ensure that our perturbation works well.

Chapter 3

Multi Fluid System Using Gauge Field Theory Approach

This chapter is the main part of the thesis. We describe multi fluids system by modelling such phenomenon as scatterings of multi buch fluid fields. This can be done by defining the amplitude of such interactions using the Navier-stokes lagrangian developed in the preceeding chapter in a similar manner as in the elementary particle physics.

3.1 Feyman Diagram for Fluid System

In the fluid dynamics which is governed by the NS equation we are mostly interested only in how the forces are mediated, and not in the transition of an initial state to another final state as concerned in particle physics. Within this interest, we need to consider only the bosonic terms in the total lagrangian. Assuming that the lagrangian is invariant under certain gauge symmetry, we have from Eq. (2.10),

$$\mathcal{L}_{\rm NS} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \ . \tag{3.1}$$

Expanding and writing all terms explicitely,

$$\begin{aligned}
\mathcal{L}_{\rm NS} &= -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \\
&= -\frac{1}{4} [\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu][\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} - g f^{amn} A^{m\mu} A^{n\nu}] \\
&= -\frac{1}{4} \left[(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) + 2 g f^{abc} A^b_\mu A^c_\nu (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \right. \\
&\quad \left. + g^2 f^{abc} f^{amn} A^a_\mu A^b_\nu A^{m\mu} A^{n\nu} \right]
\end{aligned} \tag{3.2}$$

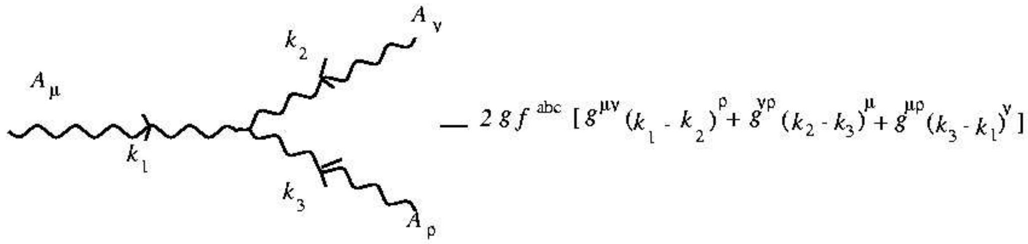

Figure 3.1: Feynman rule for three point interaction.

From these terms, we have the quadratic term which gives the propagator for the field,

$$-\frac{i}{k^2}\left[g^{\mu\nu}+(\zeta-1)\frac{k^\mu k^\nu}{k^2}\right]\delta^{ab}\ . \tag{3.3}$$

Further, the cubic and quartic terms give the interactions for three and four buch fluid fields,

$$-2gf^{abc}\left[g^{\mu\nu}(k_1-k_2)^\rho+g^{\nu\rho}(k_2-k_3)^\mu+g^{\mu\rho}(k_3-k_1)^\nu\right]\ , \tag{3.4}$$

and,

$$\begin{aligned}&-g^2\left[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\lambda}-g^{\mu\lambda}g^{\nu\rho}\right)+\right.\\ &\left.f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\lambda}-g^{\mu\rho}g^{\nu\lambda}\right)+f^{ace}f^{bde}\left(g^{\mu\lambda}g^{\nu\rho}-g^{\mu\nu}g^{\rho\lambda}\right)\right]\ .\end{aligned} \tag{3.5}$$

Clearly, Eqs. (3.3) ∼ (**??**) provide the Feynman rules for all allowed interactions in the theory as depicted in Figs. 3.1 and 3.2.

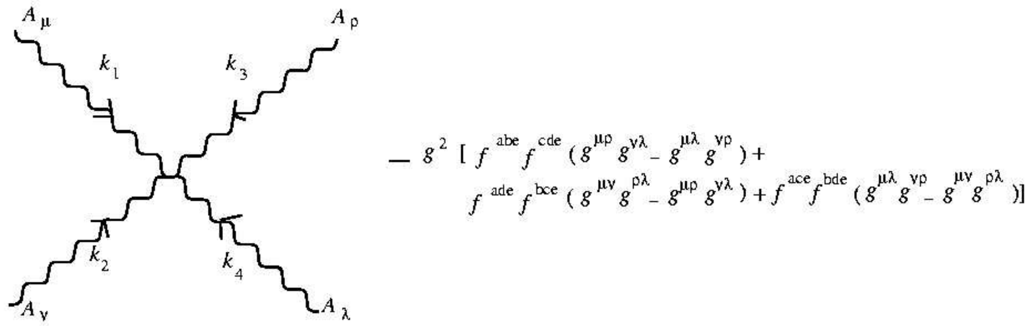

Figure 3.2: Feynman rule for four point interaction.

3.2 Multi Fluids System

In this section, we describe the dynamics of multi fluids system using the NS lagrangian. Using the allowed interactions in obtained in the previous section, we can model the hydrodynamics in term of those interactions. This means, we adopt completely the method which is familiar and used widely in the elementary particle physics. This model can be justified physically by considering that the fluid dynamics is a a result of an interaction of fluid fields which is localized at a particular point and ignoring the fluid states before interacting point. Each fluid is regarded as a separated field. Based on this scenario, we can calculate the amplitude of multi fluids system interaction, and then later on interpret and relate it with a real known observable.

As usual, we can rewrite the field in term of its polarization vector as follow,

$$A_\mu = \epsilon_\mu e^{-ik\cdot x} \quad \text{with} \quad \epsilon_\mu = \left(\frac{d}{2} |\vec{v}|^2 - V, -\vec{v} \right) , \tag{3.6}$$

where k is four-momentum. Of course, the momentum conservation still works, that is

$$\Sigma k_i = 0 \; . \tag{3.7}$$

This decomposition yields the completeness relation for the polarization vector as follow,

$$\sum_\lambda \epsilon_\mu^{\lambda\dagger} \epsilon_\nu^\lambda = \left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \right) \left(\left(\frac{d}{2} |\vec{v}|^2 - V \right)^2 - |\vec{v}|^2 \right) , \tag{3.8}$$

where the sum is over the three polarization states of massive vector fields. The proof is given in App. A. This is important to deal with the multiplication of external fields in the amplitude as done in the next section.

3.2.1 Interaction of Three Fluids System Dynamics

Having the Feynman rule at hand, we can use Eq. (3.4) to calculate the amplitude of 3 fluid fields. As written in detail in App. B we have found the transition amplitude of 3-points to be,

$$-i\mathcal{M}_3 = -2gf^{abc} \left[g^{\mu\nu}(k_1 - k_2)^\rho + g^{\nu\rho}(k_2 - k_3)^\mu + g^{\mu\rho}(k_3 - k_1)^\nu \right] A_\mu A_\nu A_\rho \; . \tag{3.9}$$

Then, this result can be immediately calculated further to obtain the squared amplitude using Eq. (3.8),

$$
\begin{aligned}
|\mathcal{M}_3|^2 \;=\;& g^2(f^{abc})^2\Big[g^{\mu\nu}(k_1-k_2)^\rho + g^{\nu\rho}(k_2-k_3)^\mu + g^{\mu\rho}(k_3-k_1)^\nu\Big] \\
& (-g_{\mu\alpha} + \frac{k_{1\mu}k_{1\alpha}}{m_1^2})(-g_{\nu\beta} + \frac{k_{2\nu}k_{2\beta}}{m_2^2})(-g_{\rho\gamma} + \frac{k_{3\rho}k_{3\gamma}}{m_3^2}) \\
& \Big[g^{\alpha\beta}(k_1-k_2)^\gamma + g^{\beta\gamma}(k_2-k_3)^\alpha + g^{\alpha\gamma}(k_3-k_1)^\beta\Big] \\
& \Big((\frac{d_1}{2}|\vec{v_1}|^2 + V_1)^2 - |\vec{v_1}|^2\Big)\Big((\frac{d_2}{2}|\vec{v_2}|^2 + V_2)^2 - |\vec{v_2}|^2\Big) \\
& \Big((\frac{d_3}{2}|\vec{v_3}|^2 + V_3)^2 - |\vec{v_3}|^2\Big)\;. \qquad (3.10)
\end{aligned}
$$

Expanding all terms, we have

$$
\begin{aligned}
|\mathcal{M}_3|^2 \;=\;& g^2(f^{abc})^2 \Big[-(k_1-k_2)^2 - 2(k_1-k_2)\cdot(k_2-k_3) - (k_2-k_3)^2 \\
& -2(k_1-k_2)\cdot(k_3-k_1) - (k_3-k_1)^2 - 2(k_2-k_3)\cdot(k_3-k_1) \\
& +\frac{k_1^2(k_1-k_2)^2}{m_1^2} + 2\frac{k_1\cdot(k_1-k_2)k_1\cdot(k_2-k_3)}{m_1^2} + \frac{(k_1\cdot(k_2-k_3))^2}{m_1^2} \\
& +2\frac{k_1\cdot(k_1-k_2)k_1\cdot(k_3-k_1)}{m_1^2} + \frac{k_1^2(k_3-k_1)^2}{m_1^2} + 2\frac{k_1\cdot(k_2-k_3)k_1\cdot(k_3-k_1)}{m_1^2} \\
& +\frac{k_2^2(k_1-k_2)^2}{m_2^2} + 2\frac{k_2\cdot(k_1-k_2)k_2\cdot(k_2-k_3)}{m_2^2} + \frac{k_2^2(k_2-k_3)^2}{m_2^2} \\
& +2\frac{k_2\cdot(k_1-k_2)k_2\cdot(k_3-k_1)}{m_2^2} + \frac{(k_2\cdot(k_3-k_1))^2}{m_2^2} + 2\frac{k_2\cdot(k_2-k_3)k_2\cdot(k_3-k_1)}{m_2^2} \\
& +\frac{(k_3\cdot(k_1-k_2))^2}{m_3^2} + 2\frac{k_3\cdot(k_1-k_2)k_3\cdot(k_2-k_3)}{m_3^2} + \frac{k_3^2(k_2-k_3)^2}{m_3^2} \\
& +2\frac{k_3\cdot(k_1-k_2)k_3\cdot(k_3-k_1)}{m_3^2} + \frac{k_3^2(k_3-k_1)^2}{m_3^2} + 2\frac{k_3\cdot(k_2-k_3)k_3\cdot(k_3-k_1)}{m_3^2} \\
& -\frac{(k_1\cdot k_2)^2(k_1-k_2)^2}{m_1^2m_2^2} - 2\frac{k_1\cdot k_2 k_1\cdot(k_2-k_3)k_2\cdot(k_1-k_2)}{m_1^2m_2^2} \\
& -\frac{k_2^2(k_1\cdot(k_2-k_3))^2}{m_1^2m_2^2} - 2\frac{k_1\cdot k_2 k_1\cdot(k_1-k_2)k_2\cdot(k_3-k_1)}{m_1^2m_2^2} \\
& -\frac{k_1^2(k_2\cdot(k_3-k_1))^2}{m_1^2m_2^2} - 2\frac{k_1\cdot k_2 k_1\cdot(k_2-k_3)k_2\cdot(k_3-k_1)}{m_1^2m_2^2} \\
& -\frac{k_1^2(k_3\cdot(k_1-k_2))^2}{m_1^2m_3^2} - 2\frac{k_1\cdot k_3 k_1\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_1^2m_3^2} \\
& -\frac{k_3^2(k_1\cdot(k_2-k_3))^2}{m_1^2m_3^2} - 2\frac{k_1\cdot k_3 k_1\cdot(k_3-k_1)k_3\cdot(k_1-k_2)}{m_1^2m_3^2} \\
& -\frac{(k_1\cdot k_3)^2(k_3-k_1)^2}{m_1^2m_3^2} - 2\frac{k_1\cdot k_3 k_1\cdot(k_2-k_3)k_3\cdot(k_3-k_1)}{m_1^2m_3^2} \\
& -\frac{k_2^2(k_3\cdot(k_1-k_2))^2}{m_2^2m_3^2} - 2\frac{k_2\cdot k_3 k_2\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_2^2m_3^2} \\
& -\frac{(k_2\cdot k_3)^2(k_2-k_3)^2}{m_2^2m_3^2} - 2\frac{k_2\cdot k_3 k_2\cdot(k_3-k_1)k_3\cdot(k_1-k_2)}{m_2^2m_3^2} \\
& -\frac{k_3^2(k_2\cdot(k_3-k_1))^2}{m_2^2m_3^2} - 2\frac{k_2\cdot k_3 k_2\cdot(k_3-k_1)k_3\cdot(k_2-k_3)}{m_2^2m_3^2} \\
& +\frac{(k_1\cdot k_2)^2(k_3\cdot(k_1-k_2))^2}{m_1^2m_2^2m_3^2} + 2\frac{k_1\cdot k_2 k_2\cdot k_3 k_1\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_1^2m_2^2m_3^2} \\
& +\frac{(k_2\cdot k_3)^2(k_1\cdot(k_2-k_3))^2}{m_1^2m_2^2m_3^2} + 2\frac{k_1\cdot k_2 k_1\cdot k_3 k_2\cdot(k_3-k_1)k_3\cdot(k_1-k_2)}{m_1^2m_2^2m_3^2}
\end{aligned}
$$

$$+\frac{(k_1\cdot k_3)^2(k_2\cdot(k_3-k_1))^2}{m_1^2m_2^2m_3^2}+2\frac{k_1\cdot k_3k_2\cdot k_3k_1\cdot(k_2-k_3)k_2\cdot(k_3-k_1)}{m_1^2m_2^2m_3^2}\Big]$$
$$\Big((\frac{d_1}{2}|\vec{v_1}|^2+V_1)^2-|\vec{v_1}|^2\Big)\Big((\frac{d_2}{2}|\vec{v_2}|^2+V_2)^2-|\vec{v_2}|^2\Big)\Big((\frac{d_3}{2}|\vec{v_3}|^2+V_3)^2-|\vec{v_3}|^2\Big) \tag{3.11}$$

Taking into account the momentum conservation, Eq. (3.7),

$$k_1+k_2+k_3=0\ , \tag{3.12}$$

we obtain several kinematic relations,

$$\begin{aligned}
& & k_i\cdot k_i = m_i^2 = \rho_i^2V^2;\\
k_1\cdot(k_2-k_3)^2 &= (\rho_3^2-\rho_2^2)V^2; & (k_1-k_2)^2 = (2\rho_1^2+2\rho_2^2-\rho_3^2)V^2;\\
k_2\cdot(k_3-k_1)^2 &= (\rho_1^2-\rho_3^2)V^2; & (k_2-k_3)^2 = (2\rho_2^2+2\rho_3^2-\rho_1^2)V^2;\\
k_3\cdot(k_1-k_2)^2 &= (\rho_2^2-\rho_1^2)V^2; & (k_3-k_1)^2 = (2\rho_1^2+2\rho_3^2-\rho_2^2)V^2;\\
k_1\cdot k_2 &= \frac{1}{2}(\rho_3^2-\rho_1^2-\rho_2^2)V^2; & k_1\cdot(k_1-k_2)^2 = \frac{1}{2}(3\rho_1^2+\rho_2^2-\rho_3^2)V^2;\\
k_1\cdot k_3 &= \frac{1}{2}(\rho_2^2-\rho_1^2-\rho_3^2)V^2; & k_2\cdot(k_2-k_3)^2 = \frac{1}{2}(3\rho_2^2+\rho_3^2-\rho_1^2)V^2;\\
k_2\cdot k_3 &= \frac{1}{2}(\rho_1^2-\rho_2^2-\rho_3^2)V^2); & k_3\cdot(k_3-k_1)^2 = \frac{1}{2}(3\rho_3^2+\rho_1^2-\rho_2^2)V^2;\\
& & k_1\cdot(k_3-k_1)^2 = \frac{1}{2}(\rho_2^2-\rho_3^2-3\rho_1^2)V^2;\\
& & k_2\cdot(k_1-k_2)^2 = \frac{1}{2}(\rho_3^1-\rho_1^2-3\rho_2^2)V^2;\\
& & k_3\cdot(k_2-k_3)^2 = \frac{1}{2}(\rho_1^2-\rho_2^2-3\rho_3^2)V^2;
\end{aligned} \tag{3.13}$$

3.2.2 Interaction of Four Fluids System Dynamics

Following the same procedure as done in Sec. 3.2.1, we obtain

$$-i\mathcal{M}_4 = -g^2\left[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\lambda}-g^{\mu\lambda}g^{\nu\rho}\right)+\right.$$
$$\left.f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\lambda}-g^{\mu\rho}g^{\nu\lambda}\right)+f^{ace}f^{bde}\left(g^{\mu\lambda}g^{\nu\rho}-g^{\mu\nu}g^{\rho\lambda}\right)\right]A_\mu A_\nu A_\rho A_\lambda\ , \tag{3.14}$$

for 4-point amplitude. Then, the squared amplitude becomes,

$$
\begin{aligned}
|\mathcal{M}_4|^2 \;=\; & g^4 \left\{ \left[f^{abe} f^{cde} \left(g^{\mu\rho} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\rho} \right) + \right. \right. \\
& f^{ade} f^{bce} \left(g^{\mu\nu} g^{\rho\lambda} - g^{\mu\rho} g^{\nu\lambda} \right) + f^{ace} f^{bde} \left(g^{\mu\lambda} g^{\nu\rho} - g^{\mu\nu} g^{\rho\lambda} \right) \Big] \\
& \left(-g_{\mu\alpha} + \frac{k_{1\mu} k_{1\alpha}}{m_1^2} \right) \left(-g_{\nu\beta} + \frac{k_{2\nu} k_{2\beta}}{m_2^2} \right) \\
& \left(-g_{\rho\gamma} + \frac{k_{3\rho} k_{3\gamma}}{m_3^2} \right) \left(-g_{\lambda}\sigma + \frac{k_{4\lambda} k_{4\sigma}}{m_4^2} \right) \\
& \Big[f^{abe} f^{cde} \left(g^{\alpha\gamma} g^{\beta\sigma} - g^{\alpha\sigma} g^{\beta\delta} \right) + \\
& \left. f^{ade} f^{bce} \left(g^{\alpha\beta} g^{\gamma\sigma} - g^{\alpha\gamma} g^{\beta\sigma} \right) + f^{ace} f^{bde} \left(g^{\alpha\sigma} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\sigma} \right) \Big] \right\} \\
& \left((\frac{d_1}{2} |\vec{v}_1|^2 + V_1)^2 - |\vec{v}_1|^2 \right) \left((\frac{d_2}{2} |\vec{v}_2|^2 + V_2)^2 - |\vec{v}_2|^2 \right) \\
& \left((\frac{d_3}{2} |\vec{v}_3|^2 + V_3)^2 - |\vec{v}_3|^2 \right) \left((\frac{d_4}{2} |\vec{v}_4|^2 + V_4)^2 - |\vec{v}_4|^2 \right) . \qquad (3.15)
\end{aligned}
$$

Expanding the result further, we get

$$
\begin{aligned}
|\mathcal{M}_4|^2 \;\; = \;\; & g^4 \Bigg[f^{abe} f^{cde} f^{abe} f^{cde} \Bigg(2\frac{k_1 \cdot k_1 k_2 \cdot k_2 - (k_1 \cdot k_2)^2}{m_1^2 m_2^2} + \frac{k_1 \cdot k_1 k_3 \cdot k_3 - (k_1 \cdot k_3)^2}{m_1^2 m_3^2} \\
& + \frac{k_1 \cdot k_1 k_4 \cdot k_4 - (k_1 \cdot k_4)^2}{m_1^2 m_4^2} + \frac{k_2 \cdot k_2 k_3 \cdot k_3 - (k_2 \cdot k_3)^2}{m_2^2 m_3^2} \\
& + \frac{k_2 \cdot k_2 k_4 \cdot k_4 - (k_2 \cdot k_4)^2}{m_2^2 m_4^2} + 2\frac{k_3 \cdot k_3 k_4 \cdot k_4 - (k_3 \cdot k_4)^2}{m_3^2 m_4^2} \\
& + \frac{2k_1 \cdot k_2 k_1 \cdot k_3 k_2 \cdot k_3 - (k_1 \cdot k_1 (k_2 \cdot k_3)^2 + k_2 \cdot k_2 (k_1 \cdot k_3)^2)}{m_1^2 m_2^2 m_3^2} \\
& + \frac{2k_1 \cdot k_2 k_1 \cdot k_4 k_2 \cdot k_4 - (k_1 \cdot k_1 (k_2 \cdot k_4)^2 + k_2 \cdot k_2 (k_1 \cdot k_4)^2)}{m_1^2 m_2^2 m_4^2} \\
& + \frac{2k_1 \cdot k_3 k_1 \cdot k_4 k_3 \cdot k_4 - (k_3 \cdot k_3 (k_1 \cdot k_4)^2 + k_4 \cdot k_4 (k_1 \cdot k_3)^2)}{m_1^2 m_3^2 m_4^2} \\
& + \frac{2k_2 \cdot k_3 k_2 \cdot k_4 k_3 \cdot k_4 - (k_3 \cdot k_3 (k_2 \cdot k_4)^2 + k_4 \cdot k_4 (k_2 \cdot k_3)^2)}{m_2^2 m_3^2 m_4^2} \\
& + \frac{(k_1 \cdot k_3 k_2 \cdot k_4)^2 + (k_1 \cdot k_4 k_2 \cdot k_3)^2 - 2(k_1 \cdot k_3 k_2 \cdot k_4)(k_1 \cdot k_4 k_2 \cdot k_3)}{m_1^2 m_2^2 m_3^2 m_4^2} \Bigg) \\
& + f^{ade} f^{bce} f^{ade} f^{bce} \Bigg(\frac{k_1 \cdot k_1 k_2 \cdot k_2 - (k_1 \cdot k_2)^2}{m_1^2 m_2^2} + \frac{k_1 \cdot k_1 k_3 \cdot k_3 - (k_1 \cdot k_3)^2}{m_1^2 m_3^2} \\
& + 2\frac{k_1 \cdot k_1 k_4 \cdot k_4 - (k_1 \cdot k_4)^2}{m_1^2 m_4^2} + 2\frac{k_2 \cdot k_2 k_3 \cdot k_3 - (k_2 \cdot k_3)^2}{m_2^2 m_3^2} \\
& + \frac{k_2 \cdot k_2 k_4 \cdot k_4 - (k_2 \cdot k_4)^2}{m_2^2 m_4^2} + \frac{k_3 \cdot k_3 k_4 \cdot k_4 - (k_3 \cdot k_4)^2}{m_3^2 m_4^2} \\
& + \frac{2k_1 \cdot k_2 k_1 \cdot k_3 k_2 \cdot k_3 - (k_2 \cdot k_2 (k_1 \cdot k_3)^2 + k_3 \cdot k_3 (k_1 \cdot k_2)^2)}{m_1^2 m_2^2 m_3^2} \\
& + \frac{2k_1 \cdot k_2 k_1 \cdot k_4 k_2 \cdot k_4 - (k_1 \cdot k_1 (k_2 \cdot k_4)^2 + k_4 \cdot k_4 (k_1 \cdot k_2)^2)}{m_1^2 m_2^2 m_4^2} \\
& + \frac{2k_1 \cdot k_3 k_1 \cdot k_4 k_3 \cdot k_4 - (k_1 \cdot k_1 (k_3 \cdot k_4)^2 + k_4 \cdot k_4 (k_1 \cdot k_3)^2)}{m_1^2 m_3^2 m_4^2} \\
& + \frac{2k_2 \cdot k_3 k_2 \cdot k_4 k_3 \cdot k_4 - (k_3 \cdot k_3 (k_2 \cdot k_4)^2 + k_2 \cdot k_2 (k_3 \cdot k_4)^2)}{m_2^2 m_3^2 m_4^2} \\
& + \frac{(k_1 \cdot k_2 k_3 \cdot k_4)^2 + (k_1 \cdot k_3 k_2 \cdot k_4)^2 - 2(k_1 \cdot k_2 k_3 \cdot k_4)(k_1 \cdot k_2 k_2 \cdot k_4)}{m_1^2 m_2^2 m_3^2 m_4^2} \Bigg)
\end{aligned}
$$

$$+f^{ace}f^{bde}f^{ace}f^{bce}\left(\frac{k_1\cdot k_1k_2\cdot k_2-(k_1\cdot k_2)^2}{m_1^2m_2^2}+2\frac{k_1\cdot k_1k_3\cdot k_3-(k_1\cdot k_3)^2}{m_1^2m_3^2}\right.$$

$$+\frac{k_1\cdot k_1k_4\cdot k_4-(k_1\cdot k_4)^2}{m_1^2m_4^2}+\frac{k_2\cdot k_2k_3\cdot k_3-(k_2\cdot k_3)^2}{m_2^2m_3^2}$$

$$+2\frac{k_2\cdot k_2k_4\cdot k_4-(k_2\cdot k_4)^2}{m_2^2m_4^2}+\frac{k_3\cdot k_3k_4\cdot k_4-(k_3\cdot k_4)^2}{m_3^2m_4^2}$$

$$+\frac{2k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3-(k_1\cdot k_1(k_2\cdot k_3)^2+k_3\cdot k_3(k_1\cdot k_2)^2)}{m_1^2m_2^2m_3^2}$$

$$+\frac{2k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4-(k_2\cdot k_2(k_1\cdot k_4)^2+k_4\cdot k_4(k_1\cdot k_2)^2)}{m_1^2m_2^2m_4^2}$$

$$+\frac{2k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4-(k_3\cdot k_3(k_1\cdot k_4)^2+k_1\cdot k_1(k_3\cdot k_4)^2)}{m_1^2m_3^2m_4^2}$$

$$+\frac{2k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4-(k_2\cdot k_2(k_2\cdot k_4)^2+k_4\cdot k_4(k_2\cdot k_3)^2)}{m_2^2m_3^2m_4^2}$$

$$\left.+\frac{(k_1\cdot k_2k_3\cdot k_4)^2+(k_1\cdot k_4k_2\cdot k_3)^2-2(k_1\cdot k_2k_3\cdot k_4)(k_1\cdot k_4k_2\cdot k_3)}{m_1^2m_2^2m_3^2m_4^2}\right)$$

$$+2f^{abe}f^{cde}f^{ade}f^{bce}\left(\frac{(k_1\cdot k_2)^2-k_1\cdot k_1k_2\cdot k_2}{m_1^2m_2^2}+\frac{(k_1\cdot k_4)^2-k_1\cdot k_1k_4\cdot k_4}{m_1^2m_4^2}\right.$$

$$+\frac{(k_2\cdot k_3)^2-k_2\cdot k_2k_3\cdot k_3}{m_2^2m_3^2}+\frac{(k_3\cdot k_4)^2-k_3\cdot k_3k_4\cdot k_4}{m_3^2m_4^2}$$

$$+\frac{k_2\cdot k_2(k_1\cdot k_3)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}$$

$$+\frac{k_1\cdot k_1(k_2\cdot k_4)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}$$

$$+\frac{k_4\cdot k_4(k_1\cdot k_3)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}$$

$$+\frac{k_3\cdot k_3(k_2\cdot k_4)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}$$

$$\left.+\frac{(k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4)}{m_12^2m_2^2m_3^2m_4^2}\right)$$

$$
\begin{aligned}
&+2f^{abe}f^{cde}f^{ace}f^{bde}\Bigg(\frac{(k_1\cdot k_2)^2-k_1\cdot k_1k_2\cdot k_2}{m_1^2m_2^2}+\frac{(k_1\cdot k_3)^2-k_1\cdot k_1k_3\cdot k_3}{m_1^2m_3^2}\\
&+\frac{(k_2\cdot k_4)^2-k_2\cdot k_2k_4\cdot k_4}{m_2^2m_4^2}+\frac{(k_3\cdot k_4)^2-k_3\cdot k_3k_4\cdot k_4}{m_3^2m_4^2}\\
&+\frac{k_1\cdot k_1(k_2\cdot k_3)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}\\
&+\frac{k_2\cdot k_2(k_1\cdot k_4)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}\\
&+\frac{k_3\cdot k_3(k_1\cdot k_4)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}\\
&+\frac{k_4\cdot k_4(k_2\cdot k_3)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}\\
&+\frac{(k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)(k_1\cdot k_4k_2\cdot k_3-k_1\cdot k_2k_3\cdot k_4)}{m_1^2m_2^2m_3^2m_4^2}\Bigg)\\
&+2f^{ade}f^{bce}f^{ace}f^{bde}\Bigg(\frac{(k_1\cdot k_3)^2-k_1\cdot k_1k_3\cdot k_3}{m_1^2m_3^2}+\frac{(k_1\cdot k_4)^2-k_1\cdot k_1k_4\cdot k_4}{m_1^2m_4^2}\\
&+\frac{(k_2\cdot k_3)^2-k_2\cdot k_2k_3\cdot k_3}{m_2^2m_3^2}+\frac{(k_2\cdot k_4)^2-k_2\cdot k_2k_4\cdot k_4}{m_2^2m_4^2}\\
&+\frac{k_3\cdot k_3(k_1\cdot k_2)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}\\
&+\frac{k_4\cdot k_4(k_1\cdot k_2)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}\\
&+\frac{k_1\cdot k_1(k_3\cdot k_4)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}\\
&+\frac{k_2\cdot k_2(k_3\cdot k_4)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}\\
&+\frac{(k_1\cdot k_4k_2\cdot k_3-k_1\cdot k_2k_3\cdot k_4)(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4)}{m_1^2m_2^2m_3^2m_4^2}\Bigg)\Bigg]\\
&\left((\frac{d_1}{2}|\vec{v}_1|^2+V_1)^2-|\vec{v}_1|^2\right)\left((\frac{d_2}{2}|\vec{v}_2|^2+V_2)^2-|\vec{v}_2|^2\right)\\
&\left((\frac{d_3}{2}|\vec{v}_3|^2+V_3)^2-|\vec{v}_3|^2\right)\left((\frac{d_4}{2}|\vec{v}_4|^2+V_4)^2-|\vec{v}_4|^2\right)
\end{aligned}
\tag{3.16}
$$

As a consequence of the momentum conservation,

$$
k_1+k_2+k_3+k_4=0\ , \tag{3.17}
$$

the following relations hold,

$$
\begin{aligned}
k_i \cdot k_i &= m_i^2 = \rho_i^2 V^2 \ , \\
k_1 \cdot k_2 &= \frac{1}{4}\rho_1 v_1 \rho_2 v_2 \left(v_1 v_2 - 4cos\theta\right) V^2 \ , \\
k_1 \cdot k_3 &= \frac{1}{4}\rho_1 v_1 \rho_3 v_3 \left(v_1 v_3 - 4cos\alpha\right) V^2 \ , \\
k_1 \cdot k_4 &= -\frac{1}{4}(4\rho_1^2 + \rho_1 v_1 \rho_2 v_2 \left(v_1 v_2 - 4cos\theta\right) + \rho_1 v_1 \rho_3 v_3 \left(v_1 v_3 - 4cos\alpha\right))V^2 \ , \\
k_2 \cdot k_3 &= -\frac{1}{4}\left(2(\rho_1^2 + \rho_2^2 + \rho_3^2 - \rho_4^2) + \rho_1 v_1 \rho_2 v_2 \left(v_1 v_2 - 4cos\theta\right)\right. \\
&\quad \left. + \rho_1 v_1 \rho_3 v_3 \left(v_1 v_3 - 4cos\alpha\right)\right) V^2 \ , \\
k_2 \cdot k_4 &= -\frac{1}{4}(2(\rho_1^2 + \rho_3^2 - \rho_2^2 - \rho_4^2) + \rho_1 v_1 \rho_3 v_3 \left(v_1 v_3 - 4cos\alpha\right))V^2 \ , \\
k_3 \cdot k_4 &= -\frac{1}{4}(2(\rho_1^2 + \rho_2^2 - \rho_3^2 - \rho_4^2) + \rho_1 v_1 \rho_2 v_2 \left(v_1 v_2 - 4cos\theta\right))V^2 \ . \qquad (3.18)
\end{aligned}
$$

Chapter 4

Discussion and conclusion

In this chapter we will describe an idea about application of the theory, *i.e.*the interactions of three or four fluids. Application of this theory can be used to show describe the observable in a meeting point of river and sea. This means we can model, for instance the turbuilence, at the point as one fluid coming from the river and the other two fluids fields scattered away to the sea.

The application of 4-point scattering is much wider [9]. This can be used to deal with turbulence in any fluid system, nebula for cosmology, dynamics of nano-crystal and so on. In all cases we can model all appropriate observables as a result of scattering of 4 homogeneous fields coming into a point.

Finally we have shown that interaction of multi fluids system which is localized on one fixed point could be counted based on gauge field theory. Further, there is a special characteristic in the 3- and 4-point interactions. In the case of 3-point there is no angle dependence at all, while in 4-point case the amplitude depends on two angles of the directions of external fields.

Appendix A

Massive vector polarization

In the case of massive vector particle, the wave equation for a particle of mass M for a bosonic field A_μ is written as [6],

$$(g^{\nu\mu}(\Box^2 + M^2) - \partial^\nu \partial^\mu) A_\mu = 0, \tag{A.1}$$

We can determine the inverse of the momentum space operator by solving

$$(g^{\nu\mu}(-k^2 + M^2) + k^\nu k^\mu)^{-1} = \delta^\mu_\lambda (A g^{\nu\lambda} + B k^\lambda k^\nu), \tag{A.2}$$

for A and B. The propagator, which is the quantity in brackets on the right-hand side of (A.2) multiplied by i, is found to be,

$$\frac{i(g^{\nu\mu} + k^\nu k^\mu / M^2)}{k^2 - M^2} \ . \tag{A.3}$$

We can show that the numerator is the sum over the spin states of the massive particle when it is taken on-shell, $i.e. k^2 = M^2$. We first take the divergence, ∂_ν of (A.1), which makes cancellation to find

$$M^2 \partial^\mu A_\mu = 0 \quad \text{that is,} \quad \partial^\mu A_\mu = 0, \tag{A.4}$$

Thus, for a massive vector particle, we have no choice but to take $\partial^\mu A_\mu = 0$. We should remark that it is not a gauge condition. As a consequence, the equation is reduced to be,

$$(\Box^2 + M^2) A_\mu = 0 \ , \tag{A.5}$$

with free-particle solutions,

$$A_\mu = \epsilon_\mu e^{-ik\cdot x} \quad \text{with} \quad \epsilon_\mu = \left(\frac{d}{2} |\vec{v}|^2 - V, -\vec{v} \right) \ . \tag{A.6}$$

The condition in Eq. (A.1) demands,

$$k^\mu \epsilon_\mu = 0 \ , \tag{A.7}$$

which reduces the number of independent polarization vector from four to three in a covariant fashion. For a vector particle of mass M, energy E and momentum $\vec{k}$ along the $z-$axis, the states of helicities λ can be described by polarization vectors

$$\begin{aligned} \epsilon^{\lambda=\pm 1} &= \mp \frac{(0,1,\pm i,0)}{\sqrt{2}} \left(\frac{d}{2} |\vec{v}|^2 - V, -\vec{v} \right) \ , \\ \epsilon^{\lambda=0} &= \frac{(|\mathbf{k}|\,,0,0,E)}{M} \left(\frac{d}{2} |\vec{v}|^2 - V, -\vec{v} \right) \ . \end{aligned} \tag{A.8}$$

Therefore the completeness relation is,

$$\sum_\lambda \epsilon_\mu^{\lambda\dagger} \epsilon_\nu^\lambda = \left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \right) \left(\left(\frac{d}{2} |\vec{v}|^2 - V \right)^2 - |\vec{v}|^2 \right) \ , \tag{A.9}$$

where the sum is over the three polarization states of massive vector particle.

Appendix B

Three points amplitude calculation

$$
\begin{aligned}
|\mathcal{M}_3|^2 &= g^2(f^{abc})^2\Big[g^{\mu\nu}(k_1-k_2)^\rho + g^{\nu\rho}(k_2-k_3)^\mu + g^{\mu\rho}(k_3-k_1)^\nu\Big] \\
&\quad (-g_{\mu\alpha}+\frac{k_{1\mu}k_{1\alpha}}{m_1^2})(-g_{\nu\beta}+\frac{k_{2\nu}k_{2\beta}}{m_2^2})(-g_{\rho\gamma}+\frac{k_{3\rho}k_{3\gamma}}{m_3^2}) \\
&\quad \Big[g^{\alpha\beta}(k_1-k_2)^\gamma + g^{\beta\gamma}(k_2-k_3)^\alpha + g^{\alpha\gamma}(k_3-k_1)^\beta\Big] \\
&\quad \Big((\frac{d_1}{2}|\vec{v_1}|^2+V_1)^2-|\vec{v_1}|^2\Big)\Big((\frac{d_2}{2}|\vec{v_2}|^2+V_2)^2-|\vec{v_2}|^2\Big) \\
&\quad \Big((\frac{d_3}{2}|\vec{v_3}|^2+V_3)^2-|\vec{v_3}|^2\Big) \\
&= g^2(f^{abc})^2\,[\text{left}\times(1)\times(2)\times(3)\times\text{right}\times\text{scalar}] \\
&= g^2(f^{abc})^2\,[\text{left}\times(1)\times(2)\times(3)\times(A+B+C)\times\text{scalar}] \qquad \text{(B.1)}
\end{aligned}
$$

$$
\text{left} = \Big[g^{\mu\nu}(k_1-k_2)^\rho + g^{\nu\rho}(k_2-k_3)^\mu + g^{\mu\rho}(k_3-k_1)^\nu\Big] \qquad \text{(B.2)}
$$

$$
(1) = (-g_{\mu\alpha}+\frac{k_{1\mu}k_{1\alpha}}{m_1^2}); \quad (2) = (-g_{\nu\beta}+\frac{k_{2\nu}k_{2\beta}}{m_2^2}); \quad (3) = (-g_{\rho\gamma}+\frac{k_{3\rho}k_{3\gamma}}{m_3^2}) \qquad \text{(B.3)}
$$

$$
\text{right} = (A+B+C) = \Big[g^{\alpha\beta}(k_1-k_2)^\gamma + g^{\beta\gamma}(k_2-k_3)^\alpha + g^{\alpha\gamma}(k_3-k_1)^\beta\Big] \qquad \text{(B.4)}
$$

$$
\text{scalar} = \Big((\frac{d_1}{2}|\vec{v_1}|^2+V_1)^2-|\vec{v_1}|^2\Big)\Big((\frac{d_2}{2}|\vec{v_2}|^2+V_2)^2-|\vec{v_2}|^2\Big)\Big((\frac{d_3}{2}|\vec{v_3}|^2+V_3)^2-|\vec{v_3}|^2\Big) \qquad \text{(B.5)}
$$

$$
\begin{aligned}
\text{left} \times (1) \;=\; & \Big[-g_\alpha^\nu(k_1-k_2)^\rho - g^{\nu\rho}(k_2-k_3)_\alpha - g_\alpha^\rho(k_3-k_1)^\nu \\
& + \frac{k_1{}^\nu k_{1\alpha}(k_1-k_2)^\rho}{m_1^2} + \frac{g^{\nu\rho}k_1\cdot(k_2-k_3)k_{1\alpha}}{m_1^2} + \frac{k_1^\rho k_{1\alpha}(k_3-k_1)^\nu}{m_1^2} \Big]
\end{aligned} \tag{B.6}
$$

$$
\begin{aligned}
\text{left} \times (1) \times (2) \;=\; & \Big[g_{\alpha\beta}(k_1-k_2)^\rho + g_\beta^\rho(k_2-k_3)_\alpha + g_\alpha^\rho(k_3-k_1)_\beta \\
& - \frac{k_{1\beta}k_{1\alpha}(k_1-k_2)^\rho}{m_1^2} - \frac{g_\beta^\rho k_1\cdot(k_2-k_3)k_{1\alpha}}{m_1^2} - \frac{k_1^\rho k_{1\alpha}(k_3-k_1)_\beta}{m_1^2} \\
& - \frac{k_{2\alpha}k_{2\beta}(k_1-k_2)^\rho}{m_2^2} - \frac{k_2^\rho k_{2\beta}(k_2-k_3)_\alpha}{m_2^2} - \frac{g_\alpha^\rho k_2\cdot(k_3-k_1)k_{2\beta}}{m_2^2} \\
& + \frac{k_1\cdot k_2 k_{1\alpha}k_{2\beta}(k_1-k_2)^\rho}{m_1^2m_2^2} + \frac{k_1\cdot(k_2-k_3)k_{1\alpha}k_{2\beta}k_2^\rho}{m_1^2m_2^2} \\
& + \frac{k_2\cdot(k_3-k_1)k_{1\alpha}k_{2\beta}k_1^\rho}{m_1^2m_2^2} \Big]
\end{aligned} \tag{B.7}
$$

$$
\begin{aligned}
& \text{left} \times (1) \times (2) \times (3) = \\
& \Big[-g_{\alpha\beta}(k_1-k_2)_\gamma - g_{\beta\gamma}(k_2-k_3)_\alpha - g_{\alpha\gamma}(k_3-k_1)_\beta \\
& + \frac{k_{1\beta}k_{1\alpha}(k_1-k_2)_\gamma}{m_1^2} + \frac{g_{\beta\gamma}k_1\cdot(k_2-k_3)k_{1\alpha}}{m_1^2} + \frac{k_{1\alpha}k_{1\gamma}(k_3-k_1)_\beta}{m_1^2} \\
& + \frac{k_{2\alpha}k_{2\beta}(k_1-k_2)_\gamma}{m_2^2} + \frac{k_{2\beta}k_{2\gamma}(k_2-k_3)_\alpha}{m_2^2} + \frac{g_{\alpha\gamma}k_2\cdot(k_3-k_1)k_{2\beta}}{m_2^2} \\
& - \frac{k_1\cdot k_2 k_{1\alpha}k_{2\beta}(k_1-k_2)_\gamma}{m_1^2m_2^2} - \frac{k_1\cdot(k_2-k_3)k_{1\alpha}k_{2\beta}k_{2\gamma}}{m_1^2m_2^2} - \frac{k_2\cdot(k_3-k_1)k_{1\alpha}k_{2\beta}k_{1\gamma}}{m_1^2m_2^2} \\
& + \frac{k_3\cdot(k_1-k_2)k_{3\gamma}g_{\alpha\beta}}{m_3^2} + \frac{k_{3\beta}k_{3\gamma}(k_2-k_3)_\alpha}{m_3^2} + \frac{k_{3\alpha}k_{3\gamma}(k_3-k_1)_\beta}{m_3^2} \\
& - \frac{k_3\cdot(k_1-k_2)k_{1\alpha}k_{1\beta}k_{3\gamma}}{m_1^2m_3^2} - \frac{k_1\cdot(k_2-k_3)k_{1\alpha}k_{3\beta}k_{3\gamma}}{m_1^2m_3^2} - \frac{k_1\cdot k_3 k_{1\alpha}(k_3-k_1)_\beta k_{3\gamma}}{m_1^2m_3^2} \\
& - \frac{k_3\cdot(k_1-k_2)k_{2\alpha}k_{2\beta}k_{3\gamma}}{m_2^2m_3^2} - \frac{k_2\cdot k_3(k_2-k_3)_\alpha k_{2\beta}k_{3\gamma}}{m_2^2m_3^2} - \frac{k_2\cdot(k_3-k_1)k_{3\alpha}k_{2\beta}k_{3\gamma}}{m_2^2m_3^2} \\
& + \frac{k_1\cdot k_2 k_3\cdot(k_1-k_2)k_{1\alpha}k_{2\beta}k_{3\gamma}}{m_1^2m_2^2m_3^2} + \frac{k_2\cdot k_3 k_1\cdot(k_2-k_3)k_{1\alpha}k_{2\beta}k_{3\gamma}}{m_1^2m_2^2m_3^2} \\
& + \frac{k_1\cdot k_3 k_2\cdot(k_3-k_1)k_{1\alpha}k_{2\beta}k_{3\gamma}}{m_1^2m_2^2m_3^2} \Big]
\end{aligned} \tag{B.8}
$$

$$
\begin{aligned}
&\text{left} \times (1) \times (2) \times (3) \times (A) = \\
&\Big[-(k_1-k_2)^2 - (k_1-k_2)\cdot(k_2-k_3) - (k_1-k_2)\cdot(k_3-k_1) \\
&+\frac{k_1^2(k_1-k_2)^2}{m_1^2} + \frac{k_1\cdot(k_1-k_2)k_1\cdot(k_2-k_3)}{m_1^2} + \frac{k_1\cdot(k_1-k_2)k_1\cdot(k_3-k_1)}{m_1^2} \\
&+\frac{k_2^2(k_1-k_2)^2}{m_2^2} + \frac{k_2\cdot(k_1-k_2)k_2\cdot(k_2-k_3)}{m_2^2} + \frac{k_2\cdot(k_1-k_2)k_2\cdot(k_3-k_1)}{m_2^2} \\
&+\frac{(k_3\cdot(k_1-k_2))^2}{m_3^2} + \frac{k_3\cdot(k_1-k_2)k_3\cdot(k_2-k_3)}{m_3^2} + \frac{k_3\cdot(k_1-k_2)k_3\cdot(k_3-k_1)}{m_3^2} \\
&-\frac{(k_1\cdot k_2)^2(k_1-k_2)^2}{m_1^2m_2^2} - \frac{k_1\cdot k_2k_1\cdot(k_2-k_3)k_2\cdot(k_1-k_2)}{m_1^2m_2^2} \\
&-\frac{k_1\cdot k_2k_2\cdot(k_3-k_1)k_1\cdot(k_1-k_2)}{m_1^2m_2^2} - \frac{k_1^2(k_3\cdot(k_1-k_2))^2}{m_1^2m_3^2} \\
&-\frac{k_1\cdot k_3k_1\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_1^2m_3^2} - \frac{k_1\cdot k_3k_1\cdot(k_3-k_1)k_3\cdot(k_1-k_2)}{m_1^2m_3^2} \\
&-\frac{k_2^2k_3\cdot(k_1-k_2)^2}{m_2^2m_3^2} - \frac{k_2\cdot k_3k_2\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_2^2m_3^2} \\
&-\frac{k_2\cdot k_3k_2\cdot(k_3-k_1)k_3\cdot(k_1-k_2)}{m_2^2m_3^2} + \frac{(k_1\cdot k_2)^2(k_3\cdot(k_1-k_2))^2}{m_1^2m_2^2m_3^2} \\
&+\frac{k_1\cdot k_2k_2\cdot k_3k_1\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_1^2m_2^2m_3^2} + \frac{k_1\cdot k_2k_1\cdot k_3k_2\cdot(k_3-k_1)k_3\cdot(k_1-k_2)}{m_1^2m_2^2m_3^2} \Big]
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
&\text{left} \times (1) \times (2) \times (3) \times (B) = \\
&\Big[-(k_1-k_2)\cdot(k_2-k_3)-(k_2-k_3)^2-(k_3-k_1)\cdot(k_2-k_3) \\
&+\frac{k_1\cdot(k_1-k_2)k_1\cdot(k_2-k_3)}{m_1^2}+\frac{(k_1\cdot(k_2-k_3))^2}{m_1^2}+\frac{k_1\cdot(k_2-k_3)k_1\cdot(k_3-k_1)}{m_1^2} \\
&+\frac{k_2\cdot(k_1-k_2)k_2\cdot(k_2-k_3)}{m_2^2}+\frac{k_2^2(k_2\cdot(k_2-k_3))^2}{m_2^2}+\frac{k_2\cdot(k_2-k_3)k_2\cdot(k_3-k_1)}{m_2^2} \\
&+\frac{k_3\cdot(k_1-k_2)k_3\cdot(k_2-k_3)}{m_3^2}+\frac{k_3^2(k_2-k_3)^2}{m_3^2}+\frac{k_3\cdot(k_2-k_3)k_3\cdot(k_3-k_1)}{m_3^2} \\
&-\frac{k_1\cdot k_2k_1\cdot(k_2-k_3)k_2\cdot(k_1-k_2)}{m_1^2m_2^2}-\frac{k_2^2(k_1\cdot(k_2-k_3))^2}{m_1^2m_2^2} \\
&-\frac{k_1\cdot k_2k_1\cdot(k_2-k_3)k_2\cdot(k_3-k_1)}{m_1^2m_2^2}-\frac{k_1\cdot k_3k_1\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_1^2m_3^2} \\
&-\frac{k_3^2(k_1\cdot(k_2-k_3))^2}{m_1^2m_3^2}-\frac{k_1\cdot k_3k_1\cdot(k_2-k_3)k_3\cdot(k_3-k_1)}{m_1^2m_3^2} \\
&-\frac{k_2\cdot k_3k_2\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_2^2m_3^2}-\frac{(k_2\cdot k_3)^2(k_2-k_3)^2}{m_2^2m_3^2} \\
&-\frac{k_2\cdot k_3k_2\cdot(k_3-k_1)k_3\cdot(k_2-k_3)}{m_2^2m_3^2}+\frac{k_1\cdot k_2k_2\cdot k_3k_1\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_1^2m_2^2m_3^2} \\
&+\frac{(k_1\cdot(k_2-k_3))^2(k_2\cdot k_3)^2}{m_1^2m_2^2m_3^2}+\frac{k_1\cdot k_3k_2\cdot k_3k_1\cdot(k_2-k_3k_2\cdot(k_3-k_1))}{m_1^2m_2^2m_3^2}\Big]
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
&\text{left} \times (1) \times (2) \times (3) \times (C) = \\
&\Big[-(k_1 - k_2) \cdot (k_3 - k_1) - (k_2 - k_3) \cdot (k_3 - k_1) - (k_3 - k_1)^2 \\
&+ \frac{k_1 \cdot (k_1 - k_2) k_1 \cdot (k_3 - k_1)}{m_1^2} + \frac{k_1 \cdot (k_2 - k_3) k_1 \cdot (k_3 - k_1)}{m_1^2} + \frac{k_1^2 (k_3 - k_1)^2}{m_1^2} \\
&+ \frac{k_2 \cdot (k_1 - k_2) k_2 \cdot (k_3 - k_1)}{m_2^2} + \frac{k_2 \cdot (k_2 - k_3) k_2 \cdot (k_3 - k_1)}{m_2^2} + \frac{(k_2 \cdot (k_3 - k_1))^2}{m_2^2} \\
&+ \frac{k_3 \cdot (k_1 - k_2) k_3 \cdot (k_3 - k_1)}{m_3^2} + \frac{k_3 \cdot (k_2 - k_3) k_3 \cdot (k_3 - k_1)}{m_3^2} + \frac{k_3^2 (k_1 - k_2)^2}{m_3^2} \\
&- \frac{k_1 \cdot k_2 k_1 \cdot (k_1 - k_2) k_2 \cdot (k_3 - k_1)}{m_1^2 m_2^2} - \frac{k_1 \cdot k_2 k_1 \cdot (k_2 - k_3) k_2 \cdot (k_3 - k_1)}{m_1^2 m_2^2} \\
&- \frac{k_1^2 (k_2 \cdot (k_3 - k_1))^2}{m_1^2 m_2^2} - \frac{k_1 \cdot k_3 k_1 \cdot (k_3 - k_1) k_3 \cdot (k_1 - k_2)}{m_1^2 m_3^2} \\
&- \frac{k_1 \cdot k_3 k_1 \cdot (k_2 - k_3) k_3 \cdot (k_3 - k_1)}{m_1^2 m_3^2} - \frac{(k_1 \cdot k_3)^2 (k_3 - k_1)^2}{m_1^2 m_3^2} \\
&- \frac{k_2 \cdot k_3 k_2 \cdot (k_3 - k_1) k_3 \cdot (k_1 - k_2)}{m_2^2 m_3^2} - \frac{k_2 \cdot k_3 k_2 \cdot (k_3 - k_1) k_3 \cdot (k_2 - k_3)}{m_2^2 m_3^2} \\
&- \frac{k_3^2 k_2 \cdot (k_3 - k_1)^2}{m_2^2 m_3^2} + \frac{k_1 \cdot k_2 k_1 \cdot k_3 k_2 \cdot (k_3 - k_1) k_3 \cdot (k_1 - k_2)}{m_1^2 m_2^2 m_3^2} \\
&+ \frac{k_1 \cdot k_3 k_2 \cdot k_3 k_1 \cdot (k_2 - k_3) k_2 \cdot (k_3 - k_1)}{m_1^2 m_2^2 m_3^2} + \frac{(k_1 \cdot k_3)^2 (k_2 \cdot (k_3 - k_1))^2}{m_1^2 m_2^2 m_3^2} \Big]
\end{aligned}
\tag{B.11}
$$

$$\text{left} \times (1) \times (2) \times (3) \times (A+B+C) = \text{left} \times (1) \times (2) \times (3) \times \text{right} =$$

$$\Big[-(k_1-k_2)^2 - 2(k_1-k_2)\cdot(k_2-k_3) - (k_2-k_3)^2$$

$$-2(k_1-k_2)\cdot(k_3-k_1) - (k_3-k_1)^2 - 2(k_2-k_3)\cdot(k_3-k_1)$$

$$+\frac{k_1^2(k_1-k_2)^2}{m_1^2} + 2\frac{k_1\cdot(k_1-k_2)k_1\cdot(k_2-k_3)}{m_1^2} + \frac{(k_1\cdot(k_2-k_3))^2}{m_1^2}$$

$$+2\frac{k_1\cdot(k_1-k_2)k_1\cdot(k_3-k_1)}{m_1^2} + \frac{k_1^2(k_3-k_1)^2}{m_1^2} + 2\frac{k_1\cdot(k_2-k_3)k_1\cdot(k_3-k_1)}{m_1^2}$$

$$+\frac{k_2^2(k_1-k_2)^2}{m_2^2} + 2\frac{k_2\cdot(k_1-k_2)k_2\cdot(k_2-k_3)}{m_2^2} + \frac{k_2^2(k_2-k_3)^2}{m_2^2}$$

$$+2\frac{k_2\cdot(k_1-k_2)k_2\cdot(k_3-k_1)}{m_2^2} + \frac{(k_2\cdot(k_3-k_1))^2}{m_2^2} + 2\frac{k_2\cdot(k_2-k_3)k_2\cdot(k_3-k_1)}{m_2^2}$$

$$+\frac{(k_3\cdot(k_1-k_2))^2}{m_3^2} + 2\frac{k_3\cdot(k_1-k_2)k_3\cdot(k_2-k_3)}{m_3^2} + \frac{k_3^2(k_2-k_3)^2}{m_3^2}$$

$$+2\frac{k_3\cdot(k_1-k_2)k_3\cdot(k_3-k_1)}{m_3^2} + \frac{k_3^2(k_3-k_1)^2}{m_3^2} + 2\frac{k_3\cdot(k_2-k_3)k_3\cdot(k_3-k_1)}{m_3^2}$$

$$-\frac{(k_1\cdot k_2)^2(k_1-k_2)^2}{m_1^2m_2^2} - 2\frac{k_1\cdot k_2k_1\cdot(k_2-k_3)k_2\cdot(k_1-k_2)}{m_1^2m_2^2}$$

$$-\frac{k_2^2(k_1\cdot(k_2-k_3))^2}{m_1^2m_2^2} - 2\frac{k_1\cdot k_2k_1\cdot(k_1-k_2)k_2\cdot(k_3-k_1)}{m_1^2m_2^2}$$

$$-\frac{k_1^2(k_2\cdot(k_3-k_1))^2}{m_1^2m_2^2} - 2\frac{k_1\cdot k_2k_1\cdot(k_2-k_3)k_2\cdot(k_3-k_1)}{m_1^2m_2^2}$$

$$-\frac{k_1^2(k_3\cdot(k_1-k_2))^2}{m_1^2m_3^2} - 2\frac{k_1\cdot k_3k_1\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_1^2m_3^2}$$

$$-\frac{k_3^2(k_1\cdot(k_2-k_3))^2}{m_1^2m_3^2} - 2\frac{k_1\cdot k_3k_1\cdot(k_3-k_1)k_3\cdot(k_1-k_2)}{m_1^2m_3^2}$$

$$-\frac{(k_1\cdot k_3)^2(k_3-k_1)^2}{m_1^2m_3^2} - 2\frac{k_1\cdot k_3k_1\cdot(k_2-k_3)k_3\cdot(k_3-k_1)}{m_1^2m_3^2}$$

$$-\frac{k_2^2(k_3\cdot(k_1-k_2))^2}{m_2^2m_3^2} - 2\frac{k_2\cdot k_3k_2\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_2^2m_3^2}$$

$$-\frac{(k_2\cdot k_3)^2(k_2-k_3)^2}{m_2^2m_3^2} - 2\frac{k_2\cdot k_3k_2\cdot(k_3-k_1)k_3\cdot(k_1-k_2)}{m_2^2m_3^2}$$

$$-\frac{k_3^2(k_2\cdot(k_3-k_1))^2}{m_2^2m_3^2} - 2\frac{k_2\cdot k_3k_2\cdot(k_3-k_1)k_3\cdot(k_2-k_3)}{m_2^2m_3^2}$$

$$+\frac{(k_1\cdot k_2)^2(k_3\cdot(k_1-k_2))^2}{m_1^2m_2^2m_3^2} + 2\frac{k_1\cdot k_2k_2\cdot k_3k_1\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_1^2m_2^2m_3^2}$$

$$
+\frac{(k_2\cdot k_3)^2(k_1\cdot(k_2-k_3))^2}{m_1^2m_2^2m_3^2}+2\frac{k_1\cdot k_2k_1\cdot k_3k_2\cdot(k_3-k_1)k_3\cdot(k_1-k_2)}{m_1^2m_2^2m_3^2}
$$

$$
+\frac{(k_1\cdot k_3)^2(k_2\cdot(k_3-k_1))^2}{m_1^2m_2^2m_3^2}+2\frac{k_1\cdot k_3k_2\cdot k_3k_1\cdot(k_2-k_3)k_2\cdot(k_3-k_1)}{m_1^2m_2^2m_3^2}\Bigg] \quad \text{(B.12)}
$$

$$|\mathcal{M}_3|^2 = g^2(f^{abc})^2\,[\text{left}\times(1)\times(2)\times(3)\times\text{right}\times\text{scalar}] =$$

$$g^2(f^{abc})^2\Big[-(k_1-k_2)^2 - 2(k_1-k_2)\cdot(k_2-k_3) - (k_2-k_3)^2$$
$$-2(k_1-k_2)\cdot(k_3-k_1) - (k_3-k_1)^2 - 2(k_2-k_3)\cdot(k_3-k_1)$$
$$+\frac{k_1^2(k_1-k_2)^2}{m_1^2} + 2\frac{k_1\cdot(k_1-k_2)k_1\cdot(k_2-k_3)}{m_1^2} + \frac{(k_1\cdot(k_2-k_3))^2}{m_1^2}$$
$$+2\frac{k_1\cdot(k_1-k_2)k_1\cdot(k_3-k_1)}{m_1^2} + \frac{k_1^2(k_3-k_1)^2}{m_1^2} + 2\frac{k_1\cdot(k_2-k_3)k_1\cdot(k_3-k_1)}{m_1^2}$$
$$+\frac{k_2^2(k_1-k_2)^2}{m_2^2} + 2\frac{k_2\cdot(k_1-k_2)k_2\cdot(k_2-k_3)}{m_2^2} + \frac{k_2^2(k_2-k_3)^2}{m_2^2}$$
$$+2\frac{k_2\cdot(k_1-k_2)k_2\cdot(k_3-k_1)}{m_2^2} + \frac{(k_2\cdot(k_3-k_1))^2}{m_2^2} + 2\frac{k_2\cdot(k_2-k_3)k_2\cdot(k_3-k_1)}{m_2^2}$$
$$+\frac{(k_3\cdot(k_1-k_2))^2}{m_3^2} + 2\frac{k_3\cdot(k_1-k_2)k_3\cdot(k_2-k_3)}{m_3^2} + \frac{k_3^2(k_2-k_3)^2}{m_3^2}$$
$$+2\frac{k_3\cdot(k_1-k_2)k_3\cdot(k_3-k_1)}{m_3^2} + \frac{k_3^2(k_3-k_1)^2}{m_3^2} + 2\frac{k_3\cdot(k_2-k_3)k_3\cdot(k_3-k_1)}{m_3^2}$$
$$-\frac{(k_1\cdot k_2)^2(k_1-k_2)^2}{m_1^2m_2^2} - 2\frac{k_1\cdot k_2k_1\cdot(k_2-k_3)k_2\cdot(k_1-k_2)}{m_1^2m_2^2}$$
$$-\frac{k_2^2(k_1\cdot(k_2-k_3))^2}{m_1^2m_2^2} - 2\frac{k_1\cdot k_2k_1\cdot(k_1-k_2)k_2\cdot(k_3-k_1)}{m_1^2m_2^2}$$
$$-\frac{k_1^2(k_2\cdot(k_3-k_1))^2}{m_1^2m_2^2} - 2\frac{k_1\cdot k_2k_1\cdot(k_2-k_3)k_2\cdot(k_3-k_1)}{m_1^2m_2^2}$$
$$-\frac{k_1^2(k_3\cdot(k_1-k_2))^2}{m_1^2m_3^2} - 2\frac{k_1\cdot k_3k_1\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_1^2m_3^2}$$
$$-\frac{k_3^2(k_1\cdot(k_2-k_3))^2}{m_1^2m_3^2} - 2\frac{k_1\cdot k_3k_1\cdot(k_3-k_1)k_3\cdot(k_1-k_2)}{m_1^2m_3^2}$$
$$-\frac{(k_1\cdot k_3)^2(k_3-k_1)^2}{m_1^2m_3^2} - 2\frac{k_1\cdot k_3k_1\cdot(k_2-k_3)k_3\cdot(k_3-k_1)}{m_1^2m_3^2}$$
$$-\frac{k_2^2(k_3\cdot(k_1-k_2))^2}{m_2^2m_3^2} - 2\frac{k_2\cdot k_3k_2\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_2^2m_3^2}$$
$$-\frac{(k_2\cdot k_3)^2(k_2-k_3)^2}{m_2^2m_3^2} - 2\frac{k_2\cdot k_3k_2\cdot(k_3-k_1)k_3\cdot(k_1-k_2)}{m_2^2m_3^2}$$
$$-\frac{k_3^2(k_2\cdot(k_3-k_1))^2}{m_2^2m_3^2} - 2\frac{k_2\cdot k_3k_2\cdot(k_3-k_1)k_3\cdot(k_2-k_3)}{m_2^2m_3^2}$$
$$+\frac{(k_1\cdot k_2)^2(k_3\cdot(k_1-k_2))^2}{m_1^2m_2^2m_3^2} + 2\frac{k_1\cdot k_2k_2\cdot k_3k_1\cdot(k_2-k_3)k_3\cdot(k_1-k_2)}{m_1^2m_2^2m_3^2}$$
$$+\frac{(k_2\cdot k_3)^2(k_1\cdot(k_2-k_3))^2}{m_1^2m_2^2m_3^2} + 2\frac{k_1\cdot k_2k_1\cdot k_3k_2\cdot(k_3-k_1)k_3\cdot(k_1-k_2)}{m_1^2m_2^2m_3^2}$$

$$+\frac{(k_1\cdot k_3)^2(k_2\cdot(k_3-k_1))^2}{m_1^2m_2^2m_3^2}+2\frac{k_1\cdot k_3k_2\cdot k_3k_1\cdot(k_2-k_3)k_2\cdot(k_3-k_1)}{m_1^2m_2^2m_3^2}\Big]$$
$$\Big((\frac{d_1}{2}|\vec{v_1}|^2+V_1)^2-|\vec{v_1}|^2\Big)\Big((\frac{d_2}{2}|\vec{v_2}|^2+V_2)^2-|\vec{v_2}|^2\Big)\Big((\frac{d_3}{2}|\vec{v_3}|^2+V_3)^2-|\vec{v_3}|^2\Big) \tag{B.13}$$

$$k_1+k_2+k_3=0 \tag{B.14}$$

$$\begin{aligned}
& & k_i\cdot k_i=m_i^2=\rho_i^2V^2;\\
k_1\cdot(k_2-k_3)^2=(\rho_3^2-\rho_2^2)V^2; & & (k_1-k_2)^2=(2\rho_1^2+2\rho_2^2-\rho_3^2)V^2;\\
k_2\cdot(k_3-k_1)^2=(\rho_1^2-\rho_3^2)V^2; & & (k_2-k_3)^2=(2\rho_2^2+2\rho_3^2-\rho_1^2)V^2;\\
k_3\cdot(k_1-k_2)^2=(\rho_2^2-\rho_1^2)V^2; & & (k_3-k_1)^2=(2\rho_1^2+2\rho_3^2-\rho_2^2)V^2;\\
k_1\cdot k_2=\frac{1}{2}(\rho_3^2-\rho_1^2-\rho_2^2)V^2; & & k_1\cdot(k_1-k_2)^2=\frac{1}{2}(3\rho_1^2+\rho_2^2-\rho_3^2)V^2;\\
k_1\cdot k_3=\frac{1}{2}(\rho_2^2-\rho_1^2-\rho_3^2)V^2; & & k_2\cdot(k_2-k_3)^2=\frac{1}{2}(3\rho_2^2+\rho_3^2-\rho_1^2)V^2;\\
k_2\cdot k_3=\frac{1}{2}(\rho_1^2-\rho_2^2-\rho_3^2)V^2); & & k_3\cdot(k_3-k_1)^2=\frac{1}{2}(3\rho_3^2+\rho_1^2-\rho_2^2)V^2;\\
& & k_1\cdot(k_3-k_1)^2=\frac{1}{2}(\rho_2^2-\rho_3^2-3\rho_1^2)V^2;\\
& & k_2\cdot(k_1-k_2)^2=\frac{1}{2}(\rho_3^1-\rho_1^2-3\rho_2^2)V^2;\\
& & k_3\cdot(k_2-k_3)^2=\frac{1}{2}(\rho_1^2-\rho_2^2-3\rho_3^2)V^2;
\end{aligned} \tag{B.15}$$

Appendix C

Four points amplitude calculation

$$
\begin{aligned}
|\mathcal{M}_4|^2 &= g^4 \big\{ \big[f^{abe} f^{cde} \left(g^{\mu\rho} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\rho} \right) + \\
&\quad f^{ade} f^{bce} \left(g^{\mu\nu} g^{\rho\lambda} - g^{\mu\rho} g^{\nu\lambda} \right) + f^{ace} f^{bde} \left(g^{\mu\lambda} g^{\nu\rho} - g^{\mu\nu} g^{\rho\lambda} \right) \big] \\
&\quad \left(-g_{\mu\alpha} + \frac{k_{1\mu} k_{1\alpha}}{m_1^2} \right) \left(-g_{\nu\beta} + \frac{k_{2\nu} k_{2\beta}}{m_2^2} \right) \\
&\quad \left(-g_{\rho\gamma} + \frac{k_{3\rho} k_{3\gamma}}{m_3^2} \right) \left(-g_{\lambda}\sigma + \frac{k_{4\lambda} k_{4\sigma}}{m_4^2} \right) \\
&\quad \big[f^{abe} f^{cde} \left(g^{\alpha\gamma} g^{\beta\sigma} - g^{\alpha\sigma} g^{\beta\delta} \right) + \\
&\quad f^{ade} f^{bce} \left(g^{\alpha\beta} g^{\gamma\sigma} - g^{\alpha\gamma} g^{\beta\sigma} \right) + f^{ace} f^{bde} \left(g^{\alpha\sigma} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\sigma} \right) \big] \big\} \\
&\quad \left((\frac{d_1}{2} |\vec{v}_1|^2 + V_1)^2 - |\vec{v}_1|^2 \right) \left((\frac{d_2}{2} |\vec{v}_2|^2 + V_2)^2 - |\vec{v}_2|^2 \right) \\
&\quad \left((\frac{d_3}{2} |\vec{v}_3|^2 + V_3)^2 - |\vec{v}_3|^2 \right) \left((\frac{d_4}{2} |\vec{v}_4|^2 + V_4)^2 - |\vec{v}_4|^2 \right) \\
&= g^4 \left[\text{left} \times (1) \times (2) \times (3) \times (4) \times \text{right} \times \text{scalar} \right] \\
&= g^4 \left[\text{left} \times (1) \times (2) \times (3) \times (4) \times (A + B + C) \times \text{scalar} \right]
\end{aligned} \tag{C.1}
$$

$$
\begin{aligned}
\text{left} &= \big[f^{abe} f^{cde} \left(g^{\mu\rho} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\rho} \right) + \\
&\quad f^{ade} f^{bce} \left(g^{\mu\nu} g^{\rho\lambda} - g^{\mu\rho} g^{\nu\lambda} \right) + f^{ace} f^{bde} \left(g^{\mu\lambda} g^{\nu\rho} - g^{\mu\nu} g^{\rho\lambda} \right) \big]
\end{aligned} \tag{C.2}
$$

$$(1) = (-g_{\mu\alpha} + \frac{k_{1\mu}k_{1\alpha}}{m_1^2}); \qquad (2) = (-g_{\nu\beta} + \frac{k_{2\nu}k_{2\beta}}{m_2^2});$$
$$(3) = (-g_{\rho\gamma} + \frac{k_{3\rho}k_{3\gamma}}{m_3^2}); \qquad (4) = (-g_{\lambda}\sigma + \frac{k_{4\lambda}k_{4\sigma}}{m_4^2}); \tag{C.3}$$

$$\begin{aligned} \text{right} &= (A+B+C) \\ &= \left[f^{abe}f^{cde}\left(g^{\alpha\beta}g^{\gamma\sigma} - g^{\alpha\gamma}g^{\beta\sigma}\right) + f^{ade}f^{bce}\left(g^{\alpha\beta}g^{\gamma\sigma} - g^{\alpha\gamma}g^{\beta\sigma}\right)\right. \\ &\quad \left.+f^{ace}f^{bde}\left(g^{\alpha\sigma}g^{\beta\gamma} - g^{\alpha\beta}g^{\gamma\sigma}\right)\right] \end{aligned} \tag{C.4}$$

$$\begin{aligned} \text{scalar} = &\left((\frac{d_1}{2}|\vec{v}_1|^2 + V_1)^2 - |\vec{v}_1|^2\right)\left((\frac{d_2}{2}|\vec{v}_2|^2 + V_2)^2 - |\vec{v}_2|^2\right) \\ &\left((\frac{d_3}{2}|\vec{v}_3|^2 + V_3)^2 - |\vec{v}_3|^2\right)\left((\frac{d_4}{2}|\vec{v}_4|^2 + V_4)^2 - |\vec{v}_4|^2\right) \end{aligned} \tag{C.5}$$

$$\begin{aligned} &\text{left} \times (1) = \\ &\quad \left[f^{abe}f^{cde}\left(-g^{\nu\lambda}g^{\rho}_{\alpha} + g^{\nu\rho}g^{\lambda}_{\alpha} + g^{\nu\lambda}\frac{k_1^{\rho}k_{1\alpha}}{m_1^2} - g^{\nu\rho}\frac{k_1^{\lambda}k_{1\alpha}}{m_1^2}\right)\right. \\ &\quad +f^{ade}f^{bce}\left(-g^{\rho\lambda}g^{\nu}_{\alpha} + g^{\nu\lambda}g^{\rho}_{\alpha} + g^{\rho\lambda}\frac{k_1^{\nu}k_{1\alpha}}{m_1^2} - g^{\nu\lambda}\frac{k_1^{\rho}k_{1\alpha}}{m_1^2}\right) \\ &\quad \left.+f^{ace}f^{bde}\left(-g^{\nu\rho}g^{\lambda}_{\alpha} + g^{\rho\lambda}g^{\nu}_{\alpha} + g^{\nu\rho}\frac{k_1^{\lambda}k_{1\alpha}}{m_1^2} - g^{\rho\lambda}\frac{k_1^{\nu}k_{1\alpha}}{m_1^2}\right)\right] \end{aligned} \tag{C.6}$$

$$\begin{aligned} &\text{left} \times (1) \times (2) = \\ &\left[f^{abe}f^{cde}\left(g^{\rho}_{\alpha}g^{\lambda}_{\beta} - g^{\rho}_{\beta}g^{\lambda}_{\alpha} - g^{\lambda}_{\beta}\frac{k_1^{\rho}k_{1\alpha}}{m_1^2} + g^{\rho}_{\beta}\frac{k_1^{\lambda}k_{1\alpha}}{m_1^2}\right.\right. \\ &\left.-g^{\rho}_{\alpha}\frac{k_2^{\lambda}k_{2\beta}}{m_2^2} + g^{\lambda}_{\alpha}\frac{k_2^{\rho}k_{2\beta}}{m_2^2} + \frac{k_1^{\rho}k_2^{\lambda}k_{1\alpha}k_{2\beta}}{m_1^2m_2^2} - \frac{k_1^{\lambda}k_2^{\rho}k_{1\alpha}k_{2\beta}}{m_1^2m_2^2}\right) \\ &+f^{ade}f^{bce}\left(g_{\alpha\beta}g^{\rho\lambda} - g^{\rho}_{\alpha}g^{\lambda}_{\beta} - g^{\rho\lambda}\frac{k_{1\beta}k_{1\alpha}}{m_1^2} + g^{\lambda}_{\beta}\frac{k_1^{\rho}k_{1\alpha}}{m_1^2}\right. \\ &\left.-g^{\rho\lambda}\frac{k_{2\alpha}k_{2\beta}}{m_2^2} + g^{\rho}_{\alpha}\frac{k_2^{\lambda}k_{2\beta}}{m_2^2} + g^{\rho\lambda}\frac{k_1\cdot k_2k_{1\alpha}k_{2\beta}}{m_1^2m_2^2} - \frac{k_1^{\rho}k_2^{\lambda}k_{1\alpha}k_{2\beta}}{m_1^2m_2^2}\right) \\ &f^{ace}f^{bde}\left(g^{\rho}_{\beta}g^{\lambda}_{\alpha} - g_{\alpha\beta}g^{\rho\lambda} - g^{\rho}_{\beta}\frac{k_1^{\lambda}k_{1\alpha}}{m_1^2} + g^{\rho\lambda}\frac{k_{1\beta}k_{1\alpha}}{m_1^2}\right. \\ &\left.\left.-g^{\lambda}_{\alpha}\frac{k_2^{\rho}k_{2\beta}}{m_2^2} + g^{\rho\lambda}\frac{k_{2\alpha}k_{2\beta}}{m_2^2} + \frac{k_1^{\lambda}k_2^{\rho}k_{1\alpha}k_{2\beta}}{m_1^2m_2^2} - g^{\rho\lambda}\frac{k_1\cdot k_2k_{1\alpha}k_{2\beta}}{m_1^2m_2^2}\right)\right] \end{aligned} \tag{C.7}$$

left $\times$ (1) $\times$ (2) $\times$ (3) $=$

$$
\begin{aligned}
&\Bigg[f^{abe}f^{cde}\Bigg(-g^{\lambda}_{\beta}g_{\alpha\gamma}+g_{\beta\gamma}g^{\lambda}_{\alpha}+g^{\lambda}_{\beta}\frac{k_{1\alpha}k_{1\gamma}}{m_1^2}-g_{\beta\gamma}\frac{k_1^{\lambda}k_{1\alpha}}{m_1^2}\\
&+g_{\alpha\gamma}\frac{k_2^{\lambda}k_{2\beta}}{m_2^2}-g^{\lambda}_{\alpha}\frac{k_{2\beta}k_{2\gamma}}{m_2^2}-\frac{k_2^{\lambda}k_{1\alpha}k_{2\beta}k_{1\gamma}}{m_1^2m_2^2}+\frac{k_1^{\lambda}k_{1\alpha}k_{2\beta}k_{2\gamma}}{m_1^2m_2^2}\\
&+g^{\lambda}_{\beta}\frac{k_{3\alpha}k_{3\gamma}}{m_3^2}-g^{\lambda}_{\alpha}\frac{k_{3\beta}k_{3\gamma}}{m_3^2}-g^{\lambda}_{\beta}\frac{k_1\cdot k_3k_{1\alpha}k_{3\gamma}}{m_1^2m_3^2}+\frac{k_1^{\lambda}k_{1\alpha}k_{3\beta}k_{3\gamma}}{m_1^2m_3^2}\\
&-\frac{k_2^{\lambda}k_{3\alpha}k_{2\beta}k_{3\gamma}}{m_2^2m_3^2}+g^{\lambda}_{\alpha}\frac{k_2\cdot k_3k_{2\beta}k_{3\gamma}}{m_2^2m_3^2}+\frac{k_1\cdot k_3k_2^{\lambda}k_{1\alpha}k_{2\beta}k_{3\gamma}}{m_1^2m_2^2m_3^2}-\frac{k_2\cdot k_3k_1^{\lambda}k_{1\alpha}k_{2\beta}k_{3\gamma}}{m_1^2m_2^2m_3^2}\Bigg)\\
&+f^{ade}f^{bce}\Bigg(-g^{\lambda}_{\gamma}g_{\alpha\beta}+g^{\lambda}_{\beta}g_{\alpha\gamma}+g^{\lambda}_{\gamma}\frac{k_{1\alpha}k_{1\beta}}{m_1^2}-g^{\lambda}_{\beta}\frac{k_{1\alpha}k_{1\gamma}}{m_1^2}\\
&+g^{\lambda}_{\gamma}\frac{k_{2\alpha}k_{2\beta}}{m_2^2}-g_{\alpha\gamma}\frac{k_2^{\lambda}k_{2\beta}}{m_2^2}-g^{\lambda}_{\gamma}\frac{k_1\cdot k_2k_{1\alpha}k_{2\beta}}{m_1^2m_2^2}+\frac{k_2^{\lambda}k_{1\alpha}k_{2\beta}k_{1\gamma}}{m_1^2m_2^2}\\
&+g_{\alpha\beta}\frac{k_3^{\lambda}k_{3\gamma}}{m_3^2}-g^{\lambda}_{\beta}\frac{k_{3\alpha}k_{3\gamma}}{m_3^2}-\frac{k_3^{\lambda}k_{1\alpha}k_{1\beta}k_{3\gamma}}{m_1^2m_3^2}+g^{\lambda}_{\beta}\frac{k_1\cdot k_3k_{1\alpha}k_{3\gamma}}{m_1^2m_3^2}\\
&-\frac{k_3^{\lambda}k_{2\alpha}k_{2\beta}k_{3\gamma}}{m_2^2m_3^2}+\frac{k_2^{\lambda}k_{3\alpha}k_{2\beta}k_{3\gamma}}{m_2^2m_3^2}+\frac{k_1\cdot k_2k_3^{\lambda}k_{1\alpha}k_{2\beta}k_{3\gamma}}{m_1^2m_2^2m_3^2}-\frac{k_1\cdot k_3k_2^{\lambda}k_{1\alpha}k_{2\beta}k_{3\gamma}}{m_1^2m_2^2m_3^2}\Bigg)\\
&+f^{ace}f^{bde}\Bigg(-g_{\beta\gamma}g^{\lambda}_{\alpha}+g^{\lambda}_{\gamma}g_{\alpha\beta}+g_{\beta\gamma}\frac{k_1^{\lambda}k_{1\alpha}}{m_1^2}-g^{\lambda}_{\gamma}\frac{k_{1\alpha}k_{1\beta}}{m_1^2}\\
&+g^{\lambda}_{\alpha}\frac{k_{2\beta}k_{2\gamma}}{m_2^2}-g^{\lambda}_{\gamma}\frac{k_{2\alpha}k_{2\beta}}{m_2^2}-\frac{k_1^{\lambda}k_{1\alpha}k_{2\beta}k_{2\gamma}}{m_1^2m_2^2}+g^{\lambda}_{\gamma}\frac{k_1\cdot k_2k_{1\alpha}k_{2\beta}}{m_1^2m_2^2}\\
&+g^{\lambda}_{\alpha}\frac{k_{3\beta}k_{3\gamma}}{m_3^2}-g_{\alpha\beta}\frac{k_3^{\lambda}k_{3\gamma}}{m_3^2}-\frac{k_1^{\lambda}k_{1\alpha}k_{3\beta}k_{3\gamma}}{m_1^2m_3^2}+\frac{k_3^{\lambda}k_{1\alpha}k_{1\beta}k_{3\gamma}}{m_1^2m_3^2}\\
&-g^{\lambda}_{\alpha}\frac{k_2\cdot k_3k_{2\beta}k_{3\gamma}}{m_2^2m_3^2}+\frac{k_3^{\lambda}k_{2\alpha}k_{2\beta}k_{3\gamma}}{m_2^2m_3^2}+\frac{k_2\cdot k_3k_1^{\lambda}k_{1\alpha}k_{2\beta}k_{3\gamma}}{m_1^2m_2^2m_3^2}-\frac{k_1\cdot k_2k_3^{\lambda}k_{1\alpha}k_{2\beta}k_{3\gamma}}{m_1^2m_2^2m_3^2}\Bigg)\Bigg]
\end{aligned}
\tag{C.8}
$$

$$
\begin{aligned}
&\text{left} \times (1) \times (2) \times (3) \times (4) = \\
&\Bigg[f^{abe} f^{cde} \Bigg(g_{\alpha\gamma} g_{\beta\sigma} - g_{\alpha\sigma} g_{\beta\gamma} - g_{\beta\sigma} \frac{k_{1\alpha} k_{1\gamma}}{m_1^2} + g_{\beta\gamma} \frac{k_{1\alpha} k_{1\sigma}}{m_1^2} \\
&- g_{\alpha\gamma} \frac{k_{2\beta} k_{2\sigma}}{m_2^2} + g_{\alpha\sigma} \frac{k_{2\beta} k_{2\gamma}}{m_2^2} + \frac{k_{1\alpha} k_{2\beta} k_{1\gamma} k_{2\sigma}}{m_1^2 m_2^2} - \frac{k_{1\alpha} k_{2\beta} k_{2\gamma} k_{1\sigma}}{m_1^2 m_2^2} \\
&- g_{\beta\sigma} \frac{k_{3\alpha} k_{3\gamma}}{m_3^2} + g_{\alpha\sigma} \frac{k_{3\beta} k_{3\gamma}}{m_3^2} + g_{\beta\sigma} \frac{k_1 \cdot k_3 k_{1\alpha} k_{3\gamma}}{m_1^2 m_3^2} - \frac{k_{1\alpha} k_{3\beta} k_{3\gamma} k_{1\sigma}}{m_1^2 m_3^2} \\
&+ \frac{k_{3\alpha} k_{2\beta} k_{3\gamma} k_{2\sigma}}{m_2^2 m_3^2} - g_{\alpha\sigma} \frac{k_2 \cdot k_3 k_{2\beta} k_{3\gamma}}{m_2^2 m_3^2} - \frac{k_1 \cdot k_3 k_{1\alpha} k_{2\beta} k_{3\gamma} k_{2\sigma}}{m_1^2 m_2^2 m_3^2} + \frac{k_2 \cdot k_3 k_{1\alpha} k_{2\beta} k_{3\gamma} k_{1\sigma}}{m_1^2 m_2^2 m_3^2} \\
&- g_{\alpha\gamma} \frac{k_{4\beta} k_{4\sigma}}{m_4^2} + g_{\beta\gamma} \frac{k_{4\alpha} k_{4\sigma}}{m_4^2} + \frac{k_{1\alpha} k_{4\beta} k_{1\gamma} k_{4\sigma}}{m_1^2 m_4^2} - g_{\beta\gamma} \frac{k_1 \cdot k_4 k_{1\alpha} k_{4\sigma}}{m_1^2 m_4^2} \\
&+ g_{\alpha\gamma} \frac{k_2 \cdot k_4 k_{2\beta} k_{4\sigma}}{m_2^2 m_4^2} - \frac{k_{4\alpha} k_{2\beta} k_{2\gamma} k_{4\sigma}}{m_2^2 m_4^2} - \frac{k_2 \cdot k_4 k_{1\alpha} k_{2\beta} k_{1\gamma} k_{4\sigma}}{m_1^2 m_2^2 m_4^2} + \frac{k_1 \cdot k_4 k_{1\alpha} k_{2\beta} k_{2\gamma} k_{4\sigma}}{m_1^2 m_2^2 m_4^2} \\
&+ \frac{k_{3\alpha} k_{4\beta} k_{3\gamma} k_{4\sigma}}{m_3^2 m_4^2} - \frac{k_{4\alpha} k_{3\beta} k_{3\gamma} k_{4\sigma}}{m_3^2 m_4^2} - \frac{k_1 \cdot k_3 k_{1\alpha} k_{4\beta} k_{3\gamma} k_{4\sigma}}{m_1^2 m_3^2 m_4^2} + \frac{k_1 \cdot k_4 k_{1\alpha} k_{3\beta} k_{3\gamma} k_{4\sigma}}{m_1^2 m_3^2 m_4^2} \\
&- \frac{k_2 \cdot k_4 k_{3\alpha} k_{2\beta} k_{3\gamma} k_{4\sigma}}{m_2^2 m_3^2 m_4^2} + \frac{k_2 \cdot k_3 k_{4\alpha} k_{2\beta} k_{3\gamma} k_{4\sigma}}{m_2^2 m_3^2 m_4^2} \\
&+ \frac{k_1 \cdot k_3 k_2 \cdot k_4 k_{1\alpha} k_{2\beta} k_{3\gamma} k_{4\sigma}}{m_1^2 m_2^2 m_3^2 m_4^2} - \frac{k_1 \cdot k_4 k_2 \cdot k_3 k_{1\alpha} k_{2\beta} k_{3\gamma} k_{4\sigma}}{m_1^2 m_2^2 m_3^2 m_4^2} \Bigg) \\
&+ f^{ade} f^{bce} \Bigg(g_{\alpha\beta} g_{\gamma\sigma} - g_{\alpha\sigma} g_{\beta\gamma} - g_{\gamma\sigma} \frac{k_{1\alpha} k_{1\beta}}{m_1^2} + g_{\beta\sigma} \frac{k_{1\alpha} k_{1\gamma}}{m_1^2} \\
&- g_{\gamma\sigma} \frac{k_{2\alpha} k_{2\beta}}{m_2^2} + g_{\alpha\gamma} \frac{k_{2\beta} k_{2\sigma}}{m_2^2} + g_{\gamma\sigma} \frac{k_1 \cdot k_2 k_{1\alpha} k_{2\beta}}{m_1^2 m_2^2} - \frac{k_{1\alpha} k_{2\beta} k_{1\gamma} k_{2\sigma}}{m_1^2 m_2^2} \\
&- g_{\alpha\beta} \frac{k_{3\gamma} k_{3\sigma}}{m_3^2} + g_{\beta\sigma} \frac{k_{3\alpha} k_{3\gamma}}{m_3^2} + \frac{k_{1\alpha} k_{1\beta} k_{3\gamma} k_{3\sigma}}{m_1^2 m_3^2} - g_{\beta\sigma} \frac{k_1 \cdot k_3 k_{1\alpha} k_{3\gamma}}{m_1^2 m_3^2} \\
&+ \frac{k_{2\alpha} k_{2\beta} k_{3\gamma} k_{3\sigma}}{m_2^2 m_3^2} - \frac{k_{3\alpha} k_{2\beta} k_{3\gamma} k_{2\sigma}}{m_2^2 m_3^2} - \frac{k_1 \cdot k_2 k_{1\alpha} k_{2\beta} k_{3\gamma} k_{3\sigma}}{m_1^2 m_2^2 m_3^2} + \frac{k_1 \cdot k_3 k_{1\alpha} k_{2\beta} k_{3\gamma} k_{2\sigma}}{m_1^2 m_2^2 m_3^2} \\
&- g_{\alpha\beta} \frac{k_{4\gamma} k_{4\sigma}}{m_4^2} + g_{\alpha\gamma} \frac{k_{4\beta} k_{4\sigma}}{m_4^2} + \frac{k_{1\alpha} k_{1\beta} k_{4\gamma} k_{4\sigma}}{m_1^2 m_4^2} - \frac{k_{1\alpha} k_{4\beta} k_{1\gamma} k_{4\sigma}}{m_1^2 m_4^2} \\
&+ \frac{k_{2\alpha} k_{2\beta} k_{4\gamma} k_{4\sigma}}{m_2^2 m_4^2} - g_{\alpha\gamma} \frac{k_2 \cdot k_4 k_{2\beta} k_{4\sigma}}{m_2^2 m_4^2} - \frac{k_1 \cdot k_2 k_{1\alpha} k_{2\beta} k_{4\gamma} k_{4\sigma}}{m_1^2 m_2^2 m_4^2} + \frac{k_2 \cdot k_4 k_{1\alpha} k_{2\beta} k_{1\gamma} k_{4\sigma}}{m_1^2 m_2^2 m_4^2} \\
&+ g_{\alpha\beta} \frac{k_3 \cdot k_4 k_3 \gamma k_{4\sigma}}{m_3^2 m_4^2} - \frac{k_{3\alpha} k_{4\beta} k_{3\gamma} k_{4\sigma}}{m_3^2 m_4^2} - \frac{k_3 \cdot k_4 k_{1\alpha} k_{1\beta} k_{3\gamma} k_{4\sigma}}{m_1^2 m_3^2 m_4^2} + \frac{k_1 \cdot k_3 k_{1\alpha} k_{4\beta} k_{3\gamma} k_{4\sigma}}{m_1^2 m_3^2 m_4^2} \\
&- \frac{k_3 \cdot k_4 k_{2\alpha} k_{2\beta} k_{3\gamma} k_{4\sigma}}{m_2^2 m_3^2 m_4^2} + \frac{k_2 \cdot k_4 k_{3\alpha} k_{2\beta} k_{3\gamma} k_{4\sigma}}{m_2^2 m_3^2 m_4^2} \\
&+ \frac{k_1 \cdot k_2 k_3 \cdot k_4 k_{1\alpha} k_{2\beta} k_{3\gamma} k_{4\sigma}}{m_1^2 m_2^2 m_3^2 m_4^2} - \frac{k_1 \cdot k_3 k_2 \cdot k_4 k_{1\alpha} k_{2\beta} k_{3\gamma} k_{4\sigma}}{m_1^2 m_2^2 m_3^2 m_4^2} \Bigg) \\
&f^{ace} f^{bde} \Bigg(g_{\alpha\sigma} g_{\beta\gamma} - g_{\alpha\beta} g_{\gamma\sigma} - g_{\beta\gamma} \frac{k_{1\alpha} k_{1\sigma}}{m_1^2} + g_{\gamma\sigma} \frac{k_{1\alpha} k_{1\beta}}{m_1^2}
\end{aligned}
\tag{C.9}
$$

$$
\begin{aligned}
&-g_{\alpha\sigma}\frac{k_{2\beta}k_{2\gamma}}{m_2^2}+g_{\gamma\sigma}\frac{k_{2\alpha}k_{2\beta}}{m_2^2}+\frac{k_{1\alpha}k_{2\beta}k_{2\gamma}k_{1\sigma}}{m_1^2m_2^2}-g_{\gamma\sigma}\frac{k_1\cdot k_2k_{1\alpha}k_{2\beta}}{m_1^2m_2^2}\\
&-g_{\alpha\sigma}\frac{k_{3\beta}k_{3\gamma}}{m_3^2}+g_{\alpha\beta}\frac{k_{3\gamma}k_{3\sigma}}{m_3^2}+\frac{k_{1\alpha}k_{3\beta}k_{3\gamma}k_{1\sigma}}{m_1^2m_3^2}-\frac{k_{1\alpha}k_{1\beta}k_{3\gamma}k_{3\sigma}}{m_1^2m_3^2}\\
&+g_{\alpha\sigma}\frac{k_2\cdot k_3k_{2\beta}k_{3\gamma}}{m_2^2m_3^2}-\frac{k_{2\alpha}k_{2\beta}k_{3\gamma}k_{3\sigma}}{m_2^2m_3^2}-\frac{k_2\cdot k_3k_{1\alpha}k_{2\beta}k_{3\gamma}k_{1\sigma}}{m_1^2m_2^2m_3^2}+\frac{k_1\cdot k_2k_{1\alpha}k_{2\beta}k_{3\gamma}k_{3\sigma}}{m_1^2m_2^2m_3^2}\\
&-g_{\beta\gamma}\frac{k_{4\alpha}k_{4\sigma}}{m_4^2}+g_{\alpha\beta}\frac{k_{4\gamma}k_{4\sigma}}{m_4^2}+g_{\beta\gamma}\frac{k_1\cdot k_4k_{1\alpha}k_{4\sigma}}{m_1^2m_4^2}-\frac{k_{1\alpha}k_{1\beta}k_{4\gamma}k_{4\sigma}}{m_1^2m_4^2}\\
&+\frac{k_{4\alpha}k_{2\beta}k_{2\gamma}k_{4\sigma}}{m_2^2m_4^2}-\frac{k_{2\alpha}k_{2\beta}k_{4\gamma}k_{4\sigma}}{m_2^2m_4^2}-\frac{k_1\cdot k_4k_{1\alpha}k_{2\beta}k_{2\gamma}k_{4\sigma}}{m_1^2m_2^2m_4^2}+\frac{k_1\cdot k_2k_{1\alpha}k_{2\beta}k_{4\gamma}k_{4\sigma}}{m_1^2m_2^2m_4^2}\\
&+\frac{k_{4\alpha}k_{3\beta}k_{3\gamma}k_{4\sigma}}{m_3^2m_4^2}-g_{\alpha\beta}\frac{k_3\cdot k_4k_{3\gamma}k_{4\sigma}}{m_3^2m_4^2}-\frac{k_1\cdot k_4k_{1\alpha}k_{3\beta}k_{3\gamma}k_{4\sigma}}{m_1^2m_3^2m_4^2}+\frac{k_3\cdot k_4k_{1\alpha}k_{1\beta}k_{3\gamma}k_{4\sigma}}{m_1^2m_3^2m_4^2}\\
&-\frac{k_2\cdot k_3k_{4\alpha}k_{2\beta}k_{3\gamma}k_{4\sigma}}{m_2^2m_3^2m_4^2}+\frac{k_3\cdot k_4k_{2\alpha}k_{2\beta}k_{3\gamma}k_{4\sigma}}{m_2^2m_3^2m_4^2}\\
&+\frac{k_1\cdot k_4k_2\cdot k_3k_{1\alpha}k_{2\beta}k_{3\gamma}k_{4\sigma}}{m_1^2m_2^2m_3^2m_4^2}-\frac{k_1\cdot k_2k_3\cdot k_4k_{1\alpha}k_{2\beta}k_{3\gamma}k_{4\sigma}}{m_1^2m_2^2m_3^2m_4^2}\Bigg)\Bigg]
\end{aligned}
\tag{C.10}
$$

left $\times$ (1) $\times$ (2) $\times$ (3) $\times$ (4) $\times$ (A) =

$$\Bigg[f^{abe}f^{cde}f^{abe}f^{cde}\Bigg(2\frac{k_1\cdot k_1 k_2\cdot k_2-(k_1\cdot k_2)^2}{m_1^2m_2^2}+\frac{k_1\cdot k_1 k_3\cdot k_3-(k_1\cdot k_3)^2}{m_1^2m_3^2}$$
$$+\frac{k_1\cdot k_1 k_4\cdot k_4-(k_1\cdot k_4)^2}{m_1^2m_4^2}+\frac{k_2\cdot k_2 k_3\cdot k_3-(k_2\cdot k_3)^2}{m_2^2m_3^2}$$
$$+\frac{k_2\cdot k_2 k_4\cdot k_4-(k_2\cdot k_4)^2}{m_2^2m_4^2}+2\frac{k_3\cdot k_3 k_4\cdot k_4-(k_3\cdot k_4)^2}{m_3^2m_4^2}$$
$$+\frac{2k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3-(k_1\cdot k_1(k_2\cdot k_3)^2+k_2\cdot k_2(k_1\cdot k_3)^2)}{m_1^2m_2^2m_3^2}$$
$$+\frac{2k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4-(k_1\cdot k_1(k_2\cdot k_4)^2+k_2\cdot k_2(k_1\cdot k_4)^2)}{m_1^2m_2^2m_4^2}$$
$$+\frac{2k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4-(k_3\cdot k_3(k_1\cdot k_4)^2+k_4\cdot k_4(k_1\cdot k_3)^2)}{m_1^2m_3^2m_4^2}$$
$$+\frac{2k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4-(k_3\cdot k_3(k_2\cdot k_4)^2+k_4\cdot k_4(k_2\cdot k_3)^2)}{m_2^2m_3^2m_4^2}$$
$$+\frac{(k_1\cdot k_3k_2\cdot k_4)^2+(k_1\cdot k_4k_2\cdot k_3)^2-2(k_1\cdot k_3k_2\cdot k_4)(k_1\cdot k_4k_2\cdot k_3)}{m_1^2m_2^2m_3^2m_4^2}\Bigg)$$
$$+f^{abe}f^{cde}f^{ade}f^{bce}\Bigg(\frac{(k_1\cdot k_2)^2-k_1\cdot k_1k_2\cdot k_2}{m_1^2m_2^2}+\frac{(k_1\cdot k_4)^2-k_1\cdot k_1k_4\cdot k_4}{m_1^2m_4^2}$$
$$+\frac{(k_2\cdot k_3)^2-k_2\cdot k_2k_3\cdot k_3}{m_2^2m_3^2}+\frac{(k_3\cdot k_4)^2-k_3\cdot k_3k_4\cdot k_4}{m_3^2m_4^2}$$
$$+\frac{k_2\cdot k_2(k_1\cdot k_3)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}$$
$$+\frac{k_1\cdot k_1(k_2\cdot k_4)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}$$
$$+\frac{k_4\cdot k_4(k_1\cdot k_3)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}$$
$$+\frac{k_3\cdot k_3(k_2\cdot k_4)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}$$
$$+\frac{(k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4)}{m_12^2m_2^2m_3^2m_4^2}\Bigg)$$
$$+f^{abe}f^{cde}f^{ace}f^{bde}\Bigg(\frac{(k_1\cdot k_2)^2-k_1\cdot k_1k_2\cdot k_2}{m_1^2m_2^2}+\frac{(k_1\cdot k_3)^2-k_1\cdot k_1k_3\cdot k_3}{m_1^2m_3^2}$$
$$+\frac{(k_2\cdot k_4)^2-k_2\cdot k_2k_4\cdot k_4}{m_2^2m_4^2}+\frac{(k_3\cdot k_4)^2-k_3\cdot k_3k_4\cdot k_4}{m_3^2m_4^2}$$

$$
\begin{aligned}
&+\frac{k_1 \cdot k_1(k_2 \cdot k_3)^2 - k_1 \cdot k_2 k_1 \cdot k_3 k_2 \cdot k_3}{m_1^2 m_2^2 m_3^2} \\
&+\frac{k_2 \cdot k_2(k_1 \cdot k_4)^2 - k_1 \cdot k_2 k_1 \cdot k_4 k_2 \cdot k_4}{m_1^2 m_2^2 m_4^2} \\
&+\frac{k_3 \cdot k_3(k_1 \cdot k_4)^2 - k_1 \cdot k_3 k_1 \cdot k_4 k_3 \cdot k_4}{m_1^2 m_3^2 m_4^2} \\
&+\frac{k_4 \cdot k_4(k_2 \cdot k_3)^2 - k_2 \cdot k_3 k_2 \cdot k_4 k_3 \cdot k_4}{m_2^2 m_3^2 m_4^2} \\
&+\frac{(k_1 \cdot k_3 k_2 \cdot k_4 - k_1 \cdot k_4 k_2 \cdot k_3)(k_1 \cdot k_4 k_2 \cdot k_3 - k_1 \cdot k_2 k_3 \cdot k_4)}{m_1^2 m_2^2 m_3^2 m_4^2}\Bigg)\Bigg] \qquad \text{(C.11)}
\end{aligned}
$$

left $\times$ (1) $\times$ (2) $\times$ (3) $\times$ (4) $\times$ (B) =

$$\Big[f^{abe}f^{cde}f^{ade}f^{bce}\Big(\frac{(k_1\cdot k_2)^2-k_1\cdot k_1k_2\cdot k_2}{m_1^2m_2^2}+\frac{(k_1\cdot k_4)^2-k_1\cdot k_1k_4\cdot k_4}{m_1^2m_4^2}$$
$$+\frac{(k_2\cdot k_3)^2-k_2\cdot k_2k_3\cdot k_3}{m_2^2m_3^2}+\frac{(k_3\cdot k_4)^2-k_3\cdot k_3k_4\cdot k_4}{m_3^2m_4^2}$$
$$+\frac{k_2\cdot k_2(k_1\cdot k_3)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}$$
$$+\frac{k_1\cdot k_1(k_2\cdot k_4)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}$$
$$+\frac{k_4\cdot k_4(k_1\cdot k_3)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}$$
$$+\frac{k_3\cdot k_3(k_2\cdot k_4)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}$$
$$+\frac{(k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4)}{m_12^2m_2^2m_3^2m_4^2}\Big)$$
$$+f^{ade}f^{bce}f^{ade}f^{bce}\Big(\frac{k_1\cdot k_1k_2\cdot k_2-(k_1\cdot k_2)^2}{m_1^2m_2^2}+\frac{k_1\cdot k_1k_3\cdot k_3-(k_1\cdot k_3)^2}{m_1^2m_3^2}$$
$$+2\frac{k_1\cdot k_1k_4\cdot k_4-(k_1\cdot k_4)^2}{m_1^2m_4^2}+2\frac{k_2\cdot k_2k_3\cdot k_3-(k_2\cdot k_3)^2}{m_2^2m_3^2}$$
$$+\frac{k_2\cdot k_2k_4\cdot k_4-(k_2\cdot k_4)^2}{m_2^2m_4^2}+\frac{k_3\cdot k_3k_4\cdot k_4-(k_3\cdot k_4)^2}{m_3^2m_4^2}$$
$$+\frac{2k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3-(k_2\cdot k_2(k_1\cdot k_3)^2+k_3\cdot k_3(k_1\cdot k_2)^2)}{m_1^2m_2^2m_3^2}$$
$$+\frac{2k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4-(k_1\cdot k_1(k_2\cdot k_4)^2+k_4\cdot k_4(k_1\cdot k_2)^2)}{m_1^2m_2^2m_4^2}$$
$$+\frac{2k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4-(k_1\cdot k_1(k_3\cdot k_4)^2+k_4\cdot k_4(k_1\cdot k_3)^2)}{m_1^2m_3^2m_4^2}$$
$$+\frac{2k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4-(k_3\cdot k_3(k_2\cdot k_4)^2+k_2\cdot k_2(k_3\cdot k_4)^2)}{m_2^2m_3^2m_4^2}$$
$$+\frac{(k_1\cdot k_2k_3\cdot k_4)^2+(k_1\cdot k_3k_2\cdot k_4)^2-2(k_1\cdot k_2k_3\cdot k_4)(k_1\cdot k_2k_2\cdot k_4)}{m_1^2m_2^2m_3^2m_4^2}\Big)$$
$$+f^{ade}f^{bce}f^{ace}f^{bde}\Big(\frac{(k_1\cdot k_3)^2-k_1\cdot k_1k_3\cdot k_3}{m_1^2m_3^2}+\frac{(k_1\cdot k_4)^2-k_1\cdot k_1k_4\cdot k_4}{m_1^2m_4^2}$$
$$+\frac{(k_2\cdot k_3)^2-k_2\cdot k_2k_3\cdot k_3}{m_2^2m_3^2}+\frac{(k_2\cdot k_4)^2-k_2\cdot k_2k_4\cdot k_4}{m_2^2m_4^2}$$
$$+\frac{k_3\cdot k_3(k_1\cdot k_2)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}$$

$$
\begin{aligned}
&+\frac{k_4 \cdot k_4 (k_1 \cdot k_2)^2 - k_1 \cdot k_2 k_1 \cdot k_4 k_2 \cdot k_4}{m_1^2 m_2^2 m_4^2} \\
&+\frac{k_1 \cdot k_1 (k_3 \cdot k_4)^2 - k_1 \cdot k_3 k_1 \cdot k_4 k_3 \cdot k_4}{m_1^2 m_3^2 m_4^2} \\
&+\frac{k_2 \cdot k_2 (k_3 \cdot k_4)^2 - k_2 \cdot k_3 k_2 \cdot k_4 k_3 \cdot k_4}{m_2^2 m_3^2 m_4^2} \\
&+\left.\left.\frac{(k_1 \cdot k_4 k_2 \cdot k_3 - k_1 \cdot k_2 k_3 \cdot k_4)(k_1 \cdot k_2 k_3 \cdot k_4 - k_1 \cdot k_3 k_2 \cdot k_4)}{m_1^2 m_2^2 m_3^2 m_4^2}\right)\right] \qquad \text{(C.12)}
\end{aligned}
$$

left $\times$ (1) $\times$ (2) $\times$ (3) $\times$ (4) $\times$ (C) =

$$\Big[f^{abe}f^{cde}f^{ace}f^{bde}\Big(\frac{(k_1\cdot k_2)^2-k_1\cdot k_1k_2\cdot k_2}{m_1^2m_2^2}+\frac{(k_1\cdot k_3)^2-k_1\cdot k_1k_3\cdot k_3}{m_1^2m_3^2}$$
$$+\frac{(k_2\cdot k_4)^2-k_2\cdot k_2k_4\cdot k_4}{m_2^2m_4^2}+\frac{(k_3\cdot k_4)^2-k_3\cdot k_3k_4\cdot k_4}{m_3^2m_4^2}$$
$$+\frac{k_1\cdot k_1(k_2\cdot k_3)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}$$
$$+\frac{k_2\cdot k_2(k_1\cdot k_4)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}$$
$$+\frac{k_3\cdot k_3(k_1\cdot k_4)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}$$
$$+\frac{k_4\cdot k_4(k_2\cdot k_3)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}$$
$$+\frac{(k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)(k_1\cdot k_4k_2\cdot k_3-k_1\cdot k_2k_3\cdot k_4)}{m_1^2m_2^2m_3^2m_4^2}\Big)$$
$$+f^{ade}f^{bce}f^{ace}f^{bde}\Big(\frac{(k_1\cdot k_3)^2-k_1\cdot k_1k_3\cdot k_3}{m_1^2m_3^2}+\frac{(k_1\cdot k_4)^2-k_1\cdot k_1k_4\cdot k_4}{m_1^2m_4^2}$$
$$+\frac{(k_2\cdot k_3)^2-k_2\cdot k_2k_3\cdot k_3}{m_2^2m_3^2}+\frac{(k_2\cdot k_4)^2-k_2\cdot k_2k_4\cdot k_4}{m_2^2m_4^2}$$
$$+\frac{k_3\cdot k_3(k_1\cdot k_2)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}$$
$$+\frac{k_4\cdot k_4(k_1\cdot k_2)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}$$
$$+\frac{k_1\cdot k_1(k_3\cdot k_4)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}$$
$$+\frac{k_2\cdot k_2(k_3\cdot k_4)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}$$
$$+\frac{(k_1\cdot k_4k_2\cdot k_3-k_1\cdot k_2k_3\cdot k_4)(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4)}{m_1^2m_2^2m_3^2m_4^2}\Big)$$
$$+f^{ace}f^{bde}f^{ace}f^{bce}\Big(\frac{k_1\cdot k_1k_2\cdot k_2-(k_1\cdot k_2)^2}{m_1^2m_2^2}+2\frac{k_1\cdot k_1k_3\cdot k_3-(k_1\cdot k_3)^2}{m_1^2m_3^2}$$
$$+\frac{k_1\cdot k_1k_4\cdot k_4-(k_1\cdot k_4)^2}{m_1^2m_4^2}+\frac{k_2\cdot k_2k_3\cdot k_3-(k_2\cdot k_3)^2}{m_2^2m_3^2}$$
$$+2\frac{k_2\cdot k_2k_4\cdot k_4-(k_2\cdot k_4)^2}{m_2^2m_4^2}+\frac{k_3\cdot k_3k_4\cdot k_4-(k_3\cdot k_4)^2}{m_3^2m_4^2}$$

$$
\begin{aligned}
&+\frac{2k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3-(k_1\cdot k_1(k_2\cdot k_3)^2+k_3\cdot k_3(k_1\cdot k_2)^2)}{m_1^2m_2^2m_3^2}\\
&+\frac{2k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4-(k_2\cdot k_2(k_1\cdot k_4)^2+k_4\cdot k_4(k_1\cdot k_2)^2)}{m_1^2m_2^2m_4^2}\\
&+\frac{2k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4-(k_3\cdot k_3(k_1\cdot k_4)^2+k_1\cdot k_1(k_3\cdot k_4)^2)}{m_1^2m_3^2m_4^2}\\
&+\frac{2k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4-(k_2\cdot k_2(k_2\cdot k_4)^2+k_4\cdot k_4(k_2\cdot k_3)^2)}{m_2^2m_3^2m_4^2}\\
&+\left.\left.\frac{(k_1\cdot k_2k_3\cdot k_4)^2+(k_1\cdot k_4k_2\cdot k_3)^2-2(k_1\cdot k_2k_3\cdot k_4)(k_1\cdot k_4k_2\cdot k_3)}{m_1^2m_2^2m_3^2m_4^2}\right)\right]
\end{aligned}
\tag{C.13}
$$

$$
\begin{aligned}
&\text{left} \times (1) \times (2) \times (3) \times (4) \times (A+B+C) = \text{left} \times (1) \times (2) \times (3) \times (4) \times \text{right} = \\
&\Big[f^{abe} f^{cde} f^{abe} f^{cde} \Big(2\frac{k_1\cdot k_1 k_2\cdot k_2 - (k_1\cdot k_2)^2}{m_1^2 m_2^2} + \frac{k_1\cdot k_1 k_3\cdot k_3 - (k_1\cdot k_3)^2}{m_1^2 m_3^2} \\
&+ \frac{k_1\cdot k_1 k_4\cdot k_4 - (k_1\cdot k_4)^2}{m_1^2 m_4^2} + \frac{k_2\cdot k_2 k_3\cdot k_3 - (k_2\cdot k_3)^2}{m_2^2 m_3^2} \\
&+ \frac{k_2\cdot k_2 k_4\cdot k_4 - (k_2\cdot k_4)^2}{m_2^2 m_4^2} + 2\frac{k_3\cdot k_3 k_4\cdot k_4 - (k_3\cdot k_4)^2}{m_3^2 m_4^2} \\
&+ \frac{2k_1\cdot k_2 k_1\cdot k_3 k_2\cdot k_3 - (k_1\cdot k_1 (k_2\cdot k_3)^2 + k_2\cdot k_2 (k_1\cdot k_3)^2)}{m_1^2 m_2^2 m_3^2} \\
&+ \frac{2k_1\cdot k_2 k_1\cdot k_4 k_2\cdot k_4 - (k_1\cdot k_1 (k_2\cdot k_4)^2 + k_2\cdot k_2 (k_1\cdot k_4)^2)}{m_1^2 m_2^2 m_4^2} \\
&+ \frac{2k_1\cdot k_3 k_1\cdot k_4 k_3\cdot k_4 - (k_3\cdot k_3 (k_1\cdot k_4)^2 + k_4\cdot k_4 (k_1\cdot k_3)^2)}{m_1^2 m_3^2 m_4^2} \\
&+ \frac{2k_2\cdot k_3 k_2\cdot k_4 k_3\cdot k_4 - (k_3\cdot k_3 (k_2\cdot k_4)^2 + k_4\cdot k_4 (k_2\cdot k_3)^2)}{m_2^2 m_3^2 m_4^2} \\
&+ \frac{(k_1\cdot k_3 k_2\cdot k_4)^2 + (k_1\cdot k_4 k_2\cdot k_3)^2 - 2(k_1\cdot k_3 k_2\cdot k_4)(k_1\cdot k_4 k_2\cdot k_3)}{m_1^2 m_2^2 m_3^2 m_4^2} \Big) \\
&+ f^{ade} f^{bce} f^{ade} f^{bce} \Big(\frac{k_1\cdot k_1 k_2\cdot k_2 - (k_1\cdot k_2)^2}{m_1^2 m_2^2} + \frac{k_1\cdot k_1 k_3\cdot k_3 - (k_1\cdot k_3)^2}{m_1^2 m_3^2} \\
&+ 2\frac{k_1\cdot k_1 k_4\cdot k_4 - (k_1\cdot k_4)^2}{m_1^2 m_4^2} + 2\frac{k_2\cdot k_2 k_3\cdot k_3 - (k_2\cdot k_3)^2}{m_2^2 m_3^2} \\
&+ \frac{k_2\cdot k_2 k_4\cdot k_4 - (k_2\cdot k_4)^2}{m_2^2 m_4^2} + \frac{k_3\cdot k_3 k_4\cdot k_4 - (k_3\cdot k_4)^2}{m_3^2 m_4^2} \\
&+ \frac{2k_1\cdot k_2 k_1\cdot k_3 k_2\cdot k_3 - (k_2\cdot k_2 (k_1\cdot k_3)^2 + k_3\cdot k_3 (k_1\cdot k_2)^2)}{m_1^2 m_2^2 m_3^2} \\
&+ \frac{2k_1\cdot k_2 k_1\cdot k_4 k_2\cdot k_4 - (k_1\cdot k_1 (k_2\cdot k_4)^2 + k_4\cdot k_4 (k_1\cdot k_2)^2)}{m_1^2 m_2^2 m_4^2} \\
&+ \frac{2k_1\cdot k_3 k_1\cdot k_4 k_3\cdot k_4 - (k_1\cdot k_1 (k_3\cdot k_4)^2 + k_4\cdot k_4 (k_1\cdot k_3)^2)}{m_1^2 m_3^2 m_4^2} \\
&+ \frac{2k_2\cdot k_3 k_2\cdot k_4 k_3\cdot k_4 - (k_3\cdot k_3 (k_2\cdot k_4)^2 + k_2\cdot k_2 (k_3\cdot k_4)^2)}{m_2^2 m_3^2 m_4^2} \\
&+ \frac{(k_1\cdot k_2 k_3\cdot k_4)^2 + (k_1\cdot k_3 k_2\cdot k_4)^2 - 2(k_1\cdot k_2 k_3\cdot k_4)(k_1\cdot k_2 k_2\cdot k_4)}{m_1^2 m_2^2 m_3^2 m_4^2} \Big) \\
&+ f^{ace} f^{bde} f^{ace} f^{bce} \Big(\frac{k_1\cdot k_1 k_2\cdot k_2 - (k_1\cdot k_2)^2}{m_1^2 m_2^2} + 2\frac{k_1\cdot k_1 k_3\cdot k_3 - (k_1\cdot k_3)^2}{m_1^2 m_3^2} \\
&+ \frac{k_1\cdot k_1 k_4\cdot k_4 - (k_1\cdot k_4)^2}{m_1^2 m_4^2} + \frac{k_2\cdot k_2 k_3\cdot k_3 - (k_2\cdot k_3)^2}{m_2^2 m_3^2} \\
&+ 2\frac{k_2\cdot k_2 k_4\cdot k_4 - (k_2\cdot k_4)^2}{m_2^2 m_4^2} + \frac{k_3\cdot k_3 k_4\cdot k_4 - (k_3\cdot k_4)^2}{m_3^2 m_4^2}
\end{aligned}
$$

$$+\frac{2k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3-(k_1\cdot k_1(k_2\cdot k_3)^2+k_3\cdot k_3(k_1\cdot k_2)^2)}{m_1^2m_2^2m_3^2}$$
$$+\frac{2k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4-(k_2\cdot k_2(k_1\cdot k_4)^2+k_4\cdot k_4(k_1\cdot k_2)^2)}{m_1^2m_2^2m_4^2}$$
$$+\frac{2k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4-(k_3\cdot k_3(k_1\cdot k_4)^2+k_1\cdot k_1(k_3\cdot k_4)^2)}{m_1^2m_3^2m_4^2}$$
$$+\frac{2k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4-(k_2\cdot k_2(k_2\cdot k_4)^2+k_4\cdot k_4(k_2\cdot k_3)^2)}{m_2^2m_3^2m_4^2}$$
$$+\frac{(k_1\cdot k_2k_3\cdot k_4)^2+(k_1\cdot k_4k_2\cdot k_3)^2-2(k_1\cdot k_2k_3\cdot k_4)(k_1\cdot k_4k_2\cdot k_3)}{m_1^2m_2^2m_3^2m_4^2}\Bigg)$$
$$+2f^{abe}f^{cde}f^{ade}f^{bce}\Bigg(\frac{(k_1\cdot k_2)^2-k_1\cdot k_1k_2\cdot k_2}{m_1^2m_2^2}+\frac{(k_1\cdot k_4)^2-k_1\cdot k_1k_4\cdot k_4}{m_1^2m_4^2}$$
$$+\frac{(k_2\cdot k_3)^2-k_2\cdot k_2k_3\cdot k_3}{m_2^2m_3^2}+\frac{(k_3\cdot k_4)^2-k_3\cdot k_3k_4\cdot k_4}{m_3^2m_4^2}$$
$$+\frac{k_2\cdot k_2(k_1\cdot k_3)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}$$
$$+\frac{k_1\cdot k_1(k_2\cdot k_4)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}$$
$$+\frac{k_4\cdot k_4(k_1\cdot k_3)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}$$
$$+\frac{k_3\cdot k_3(k_2\cdot k_4)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}$$
$$+\frac{(k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4)}{m_12^2m_2^2m_3^2m_4^2}\Bigg)$$
$$+2f^{abe}f^{cde}f^{ace}f^{bde}\Bigg(\frac{(k_1\cdot k_2)^2-k_1\cdot k_1k_2\cdot k_2}{m_1^2m_2^2}+\frac{(k_1\cdot k_3)^2-k_1\cdot k_1k_3\cdot k_3}{m_1^2m_3^2}$$
$$+\frac{(k_2\cdot k_4)^2-k_2\cdot k_2k_4\cdot k_4}{m_2^2m_4^2}+\frac{(k_3\cdot k_4)^2-k_3\cdot k_3k_4\cdot k_4}{m_3^2m_4^2}$$
$$+\frac{k_1\cdot k_1(k_2\cdot k_3)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}$$
$$+\frac{k_2\cdot k_2(k_1\cdot k_4)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}$$
$$+\frac{k_3\cdot k_3(k_1\cdot k_4)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}$$
$$+\frac{k_4\cdot k_4(k_2\cdot k_3)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}$$
$$+\frac{(k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)(k_1\cdot k_4k_2\cdot k_3-k_1\cdot k_2k_3\cdot k_4)}{m_1^2m_2^2m_3^2m_4^2}\Bigg)$$

$$
\begin{aligned}
&+2f^{ade}f^{bce}f^{ace}f^{bde}\left(\frac{(k_1\cdot k_3)^2-k_1\cdot k_1k_3\cdot k_3}{m_1^2m_3^2}+\frac{(k_1\cdot k_4)^2-k_1\cdot k_1k_4\cdot k_4}{m_1^2m_4^2}\right.\\
&+\frac{(k_2\cdot k_3)^2-k_2\cdot k_2k_3\cdot k_3}{m_2^2m_3^2}+\frac{(k_2\cdot k_4)^2-k_2\cdot k_2k_4\cdot k_4}{m_2^2m_4^2}\\
&+\frac{k_3\cdot k_3(k_1\cdot k_2)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}\\
&+\frac{k_4\cdot k_4(k_1\cdot k_2)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}\\
&+\frac{k_1\cdot k_1(k_3\cdot k_4)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}\\
&+\frac{k_2\cdot k_2(k_3\cdot k_4)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}\\
&\left.+\frac{(k_1\cdot k_4k_2\cdot k_3-k_1\cdot k_2k_3\cdot k_4)(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4)}{m_1^2m_2^2m_3^2m_4^2}\right)\Bigg]
\end{aligned}
\tag{C.14}
$$

$$
\begin{aligned}
|\mathcal{M}_4|^2 &= g^4 \Bigg[f^{abe} f^{cde} f^{abe} f^{cde} \Bigg(2\frac{k_1 \cdot k_1 k_2 \cdot k_2 - (k_1 \cdot k_2)^2}{m_1^2 m_2^2} + \frac{k_1 \cdot k_1 k_3 \cdot k_3 - (k_1 \cdot k_3)^2}{m_1^2 m_3^2} \\
&+ \frac{k_1 \cdot k_1 k_4 \cdot k_4 - (k_1 \cdot k_4)^2}{m_1^2 m_4^2} + \frac{k_2 \cdot k_2 k_3 \cdot k_3 - (k_2 \cdot k_3)^2}{m_2^2 m_3^2} \\
&+ \frac{k_2 \cdot k_2 k_4 \cdot k_4 - (k_2 \cdot k_4)^2}{m_2^2 m_4^2} + 2\frac{k_3 \cdot k_3 k_4 \cdot k_4 - (k_3 \cdot k_4)^2}{m_3^2 m_4^2} \\
&+ \frac{2 k_1 \cdot k_2 k_1 \cdot k_3 k_2 \cdot k_3 - (k_1 \cdot k_1 (k_2 \cdot k_3)^2 + k_2 \cdot k_2 (k_1 \cdot k_3)^2)}{m_1^2 m_2^2 m_3^2} \\
&+ \frac{2 k_1 \cdot k_2 k_1 \cdot k_4 k_2 \cdot k_4 - (k_1 \cdot k_1 (k_2 \cdot k_4)^2 + k_2 \cdot k_2 (k_1 \cdot k_4)^2)}{m_1^2 m_2^2 m_4^2} \\
&+ \frac{2 k_1 \cdot k_3 k_1 \cdot k_4 k_3 \cdot k_4 - (k_3 \cdot k_3 (k_1 \cdot k_4)^2 + k_4 \cdot k_4 (k_1 \cdot k_3)^2)}{m_1^2 m_3^2 m_4^2} \\
&+ \frac{2 k_2 \cdot k_3 k_2 \cdot k_4 k_3 \cdot k_4 - (k_3 \cdot k_3 (k_2 \cdot k_4)^2 + k_4 \cdot k_4 (k_2 \cdot k_3)^2)}{m_2^2 m_3^2 m_4^2} \\
&+ \frac{(k_1 \cdot k_3 k_2 \cdot k_4)^2 + (k_1 \cdot k_4 k_2 \cdot k_3)^2 - 2(k_1 \cdot k_3 k_2 \cdot k_4)(k_1 \cdot k_4 k_2 \cdot k_3)}{m_1^2 m_2^2 m_3^2 m_4^2} \Bigg) \\
&+ f^{ade} f^{bce} f^{ade} f^{bce} \Bigg(\frac{k_1 \cdot k_1 k_2 \cdot k_2 - (k_1 \cdot k_2)^2}{m_1^2 m_2^2} + \frac{k_1 \cdot k_1 k_3 \cdot k_3 - (k_1 \cdot k_3)^2}{m_1^2 m_3^2} \\
&+ 2\frac{k_1 \cdot k_1 k_4 \cdot k_4 - (k_1 \cdot k_4)^2}{m_1^2 m_4^2} + 2\frac{k_2 \cdot k_2 k_3 \cdot k_3 - (k_2 \cdot k_3)^2}{m_2^2 m_3^2} \\
&+ \frac{k_2 \cdot k_2 k_4 \cdot k_4 - (k_2 \cdot k_4)^2}{m_2^2 m_4^2} + \frac{k_3 \cdot k_3 k_4 \cdot k_4 - (k_3 \cdot k_4)^2}{m_3^2 m_4^2} \\
&+ \frac{2 k_1 \cdot k_2 k_1 \cdot k_3 k_2 \cdot k_3 - (k_2 \cdot k_2 (k_1 \cdot k_3)^2 + k_3 \cdot k_3 (k_1 \cdot k_2)^2)}{m_1^2 m_2^2 m_3^2} \\
&+ \frac{2 k_1 \cdot k_2 k_1 \cdot k_4 k_2 \cdot k_4 - (k_1 \cdot k_1 (k_2 \cdot k_4)^2 + k_4 \cdot k_4 (k_1 \cdot k_2)^2)}{m_1^2 m_2^2 m_4^2} \\
&+ \frac{2 k_1 \cdot k_3 k_1 \cdot k_4 k_3 \cdot k_4 - (k_1 \cdot k_1 (k_3 \cdot k_4)^2 + k_4 \cdot k_4 (k_1 \cdot k_3)^2)}{m_1^2 m_3^2 m_4^2} \\
&+ \frac{2 k_2 \cdot k_3 k_2 \cdot k_4 k_3 \cdot k_4 - (k_3 \cdot k_3 (k_2 \cdot k_4)^2 + k_2 \cdot k_2 (k_3 \cdot k_4)^2)}{m_2^2 m_3^2 m_4^2} \\
&+ \frac{(k_1 \cdot k_2 k_3 \cdot k_4)^2 + (k_1 \cdot k_3 k_2 \cdot k_4)^2 - 2(k_1 \cdot k_2 k_3 \cdot k_4)(k_1 \cdot k_2 k_2 \cdot k_4)}{m_1^2 m_2^2 m_3^2 m_4^2} \Bigg) \\
&+ f^{ace} f^{bde} f^{ace} f^{bce} \Bigg(\frac{k_1 \cdot k_1 k_2 \cdot k_2 - (k_1 \cdot k_2)^2}{m_1^2 m_2^2} + 2\frac{k_1 \cdot k_1 k_3 \cdot k_3 - (k_1 \cdot k_3)^2}{m_1^2 m_3^2} \\
&+ \frac{k_1 \cdot k_1 k_4 \cdot k_4 - (k_1 \cdot k_4)^2}{m_1^2 m_4^2} + \frac{k_2 \cdot k_2 k_3 \cdot k_3 - (k_2 \cdot k_3)^2}{m_2^2 m_3^2} \\
&+ 2\frac{k_2 \cdot k_2 k_4 \cdot k_4 - (k_2 \cdot k_4)^2}{m_2^2 m_4^2} + \frac{k_3 \cdot k_3 k_4 \cdot k_4 - (k_3 \cdot k_4)^2}{m_3^2 m_4^2}
\end{aligned}
$$

$$
+\frac{2k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3-(k_1\cdot k_1(k_2\cdot k_3)^2+k_3\cdot k_3(k_1\cdot k_2)^2)}{m_1^2m_2^2m_3^2}
$$
$$
+\frac{2k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4-(k_2\cdot k_2(k_1\cdot k_4)^2+k_4\cdot k_4(k_1\cdot k_2)^2)}{m_1^2m_2^2m_4^2}
$$
$$
+\frac{2k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4-(k_3\cdot k_3(k_1\cdot k_4)^2+k_1\cdot k_1(k_3\cdot k_4)^2)}{m_1^2m_3^2m_4^2}
$$
$$
+\frac{2k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4-(k_2\cdot k_2(k_2\cdot k_4)^2+k_4\cdot k_4(k_2\cdot k_3)^2)}{m_2^2m_3^2m_4^2}
$$
$$
+\frac{(k_1\cdot k_2k_3\cdot k_4)^2+(k_1\cdot k_4k_2\cdot k_3)^2-2(k_1\cdot k_2k_3\cdot k_4)(k_1\cdot k_4k_2\cdot k_3)}{m_1^2m_2^2m_3^2m_4^2}\Bigg)
$$
$$
+2f^{abe}f^{cde}f^{ade}f^{bce}\Bigg(\frac{(k_1\cdot k_2)^2-k_1\cdot k_1k_2\cdot k_2}{m_1^2m_2^2}+\frac{(k_1\cdot k_4)^2-k_1\cdot k_1k_4\cdot k_4}{m_1^2m_4^2}
$$
$$
+\frac{(k_2\cdot k_3)^2-k_2\cdot k_2k_3\cdot k_3}{m_2^2m_3^2}+\frac{(k_3\cdot k_4)^2-k_3\cdot k_3k_4\cdot k_4}{m_3^2m_4^2}
$$
$$
+\frac{k_2\cdot k_2(k_1\cdot k_3)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}
$$
$$
+\frac{k_1\cdot k_1(k_2\cdot k_4)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}
$$
$$
+\frac{k_4\cdot k_4(k_1\cdot k_3)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}
$$
$$
+\frac{k_3\cdot k_3(k_2\cdot k_4)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}
$$
$$
+\frac{(k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4)}{m_12^2m_2^2m_3^2m_4^2}\Bigg)
$$
$$
+2f^{abe}f^{cde}f^{ace}f^{bde}\Bigg(\frac{(k_1\cdot k_2)^2-k_1\cdot k_1k_2\cdot k_2}{m_1^2m_2^2}+\frac{(k_1\cdot k_3)^2-k_1\cdot k_1k_3\cdot k_3}{m_1^2m_3^2}
$$
$$
+\frac{(k_2\cdot k_4)^2-k_2\cdot k_2k_4\cdot k_4}{m_2^2m_4^2}+\frac{(k_3\cdot k_4)^2-k_3\cdot k_3k_4\cdot k_4}{m_3^2m_4^2}
$$
$$
+\frac{k_1\cdot k_1(k_2\cdot k_3)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}
$$
$$
+\frac{k_2\cdot k_2(k_1\cdot k_4)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}
$$
$$
+\frac{k_3\cdot k_3(k_1\cdot k_4)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}
$$
$$
+\frac{k_4\cdot k_4(k_2\cdot k_3)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}
$$

$$
\begin{aligned}
&+\frac{(k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)(k_1\cdot k_4k_2\cdot k_3-k_1\cdot k_2k_3\cdot k_4)}{m_1^2m_2^2m_3^2m_4^2}\Bigg)\\
&+2f^{ade}f^{bce}f^{ace}f^{bde}\Bigg(\frac{(k_1\cdot k_3)^2-k_1\cdot k_1k_3\cdot k_3}{m_1^2m_3^2}+\frac{(k_1\cdot k_4)^2-k_1\cdot k_1k_4\cdot k_4}{m_1^2m_4^2}\\
&+\frac{(k_2\cdot k_3)^2-k_2\cdot k_2k_3\cdot k_3}{m_2^2m_3^2}+\frac{(k_2\cdot k_4)^2-k_2\cdot k_2k_4\cdot k_4}{m_2^2m_4^2}\\
&+\frac{k_3\cdot k_3(k_1\cdot k_2)^2-k_1\cdot k_2k_1\cdot k_3k_2\cdot k_3}{m_1^2m_2^2m_3^2}\\
&+\frac{k_4\cdot k_4(k_1\cdot k_2)^2-k_1\cdot k_2k_1\cdot k_4k_2\cdot k_4}{m_1^2m_2^2m_4^2}\\
&+\frac{k_1\cdot k_1(k_3\cdot k_4)^2-k_1\cdot k_3k_1\cdot k_4k_3\cdot k_4}{m_1^2m_3^2m_4^2}\\
&+\frac{k_2\cdot k_2(k_3\cdot k_4)^2-k_2\cdot k_3k_2\cdot k_4k_3\cdot k_4}{m_2^2m_3^2m_4^2}\\
&+\frac{(k_1\cdot k_4k_2\cdot k_3-k_1\cdot k_2k_3\cdot k_4)(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4)}{m_1^2m_2^2m_3^2m_4^2}\Bigg)\Bigg]\\
&\left((\frac{d_1}{2}|\vec{v}_1|^2+V_1)^2-|\vec{v}_1|^2\right)\left((\frac{d_2}{2}|\vec{v}_2|^2+V_2)^2-|\vec{v}_2|^2\right)\\
&\left((\frac{d_3}{2}|\vec{v}_3|^2+V_3)^2-|\vec{v}_3|^2\right)\left((\frac{d_4}{2}|\vec{v}_4|^2+V_4)^2-|\vec{v}_4|^2\right)
\end{aligned}
\tag{C.15}
$$

$$
k_1+k_2+k_3+k_4=0 \tag{C.16}
$$

$$
\begin{aligned}
k_i\cdot k_i &= m_i^2=\rho_i^2V^2;\\
k_1\cdot k_2 &= \frac{1}{4}\rho_1v_1\rho_2v_2\left(v_1v_2-4cos\theta\right)V^2;\\
k_1\cdot k_3 &= \frac{1}{4}\rho_1v_1\rho_3v_3\left(v_1v_3-4cos\alpha\right)V^2;\\
k_1\cdot k_4 &= -\frac{1}{4}(4\rho_1^2+\rho_1v_1\rho_2v_2\left(v_1v_2-4cos\theta\right)+\rho_1v_1\rho_3v_3\left(v_1v_3-4cos\alpha\right))V^2;\\
k_2\cdot k_3 &= -\frac{1}{4}\left(2(\rho_1^2+\rho_2^2+\rho_3^2-\rho_4^2)+\rho_1v_1\rho_2v_2\left(v_1v_2-4cos\theta\right)+\rho_1v_1\rho_3v_3\left(v_1v_3-4cos\alpha\right)\right)V^2\\
k_2\cdot k_4 &= -\frac{1}{4}(2(\rho_1^2+\rho_3^2-\rho_2^2-\rho_4^2)+\rho_1v_1\rho_3v_3\left(v_1v_3-4cos\alpha\right))V^2\\
k_3\cdot k_4 &= -\frac{1}{4}(2(\rho_1^2+\rho_2^2-\rho_3^2-\rho_4^2)+\rho_1v_1\rho_2v_2\left(v_1v_2-4cos\theta\right))V^2
\end{aligned}
\tag{C.17}
$$

References

[1] A. Sulaiman, *Master Theses at Department of Physics, University of Indonesia* (2005) ;
A. Sulaiman and L.T. Handoko, *in preparation.*

[2] M.V. Altaisky and J.C. Bowman, *Acta Phys. Pol.* (in press);
A.C.R. Mendes, C. Neves, W. Oliveira and F.I. Takakura, *Brazilian J. Phys.* **33** (2003) 346;
P. Lynch, *Lecture Note : Hamiltonian Methods for Geophysical Fluid Dynamics - An Introduction* (2002).

[3] A. Grauel and W.H. Steeb, *Int. J. Theor. Phys.* **24** (1985) 255.

[4] B. Bistrovick, R. Jackiw, H. Li and V.P. Nair, *Phys.Rev.* **D67** (2003) 025013;
T. Kambe, *Fluid Dyn. Res.* **32** (2003) 192;
R. Jackiw, *Proc. of Renormalization Group and Anomalies in Gravity and Cosmology, Ouro Preto, Brazil* (2003).

[5] For example, see : T.P. Cheng and L.F. Li, *Gauge theory of elementary particle physics, Oxford Science Publications* (1991).

[6] For example, see : L.H. Ryder, *Quantum Field Theory, Cambridge University Press* (1985).

[7] For example, see : F. Stancu, *Group Theory in Subnuclear Physics, Oxford Science Publications* (1996).

[8] M.K. Verma, *Int. J. Mod. Phy.* **B15** 2001 3419.

[9] Fahd, J. Suhanjani, K.E. Saputro, and L.T. Handoko *in preparation.*