

# 2-D Numerical Reconstruction of Electrical Impedance Tomography using Tikhonov Regularization Algorithms with a-posteriori parameter choice rule. \*

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## Abstract

Electrical Impedance Tomography, as an Inverse Problem, is calculation of the resistivity distribution due to given boundary potential and current density distribution. Most of the inverse problem are ill-posed, since the measurement data are limited and imperfect. This paper describes a regularization technique for solving the ill-posed problem appeared in the inverse EIT. In this regularization technique, a smoothing function with a regularization parameter, is penalizing the objective function in order to obtain a regularized resistivity update equation. The regularization parameter can be chosen from a-posteriori information. We made comparison of 3 methods, the first method can be thought of as a discrepancy principle, where we select an initial value of the regularization parameter by trial and error technique. The second and third methods are methods adopted from Linear ill-posed problem, with a posteriori information characters. We presents numerically the reconstruction using artificially generated data.

*Keywords: Inverse Problems; Nonlinear ill-posed; Tikhonov Regularization ; Reconstruction Algorithms; a-posteriori parameter.*

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## 1 Introduction

Electrical Impedance Tomography (EIT) is a computerized tomographic imaging technique which is able to reconstruct an image of the distribution of electrical impedance such as resistivity from a knowledge of the boundary voltage and current on the object. EIT offers a possibility of realizing a low cost and safe imaging system, because it uses non-ionizing radiation and requires relatively simple hardware. Some promising fields of this technique are biomedical engineering, geophysics, non-destructive test, industrial process, humanitarian demining etc. The problem of image reconstruction in EIT is broken into a forward and inverse problem. The forward problem involves finding the potential distribution given the resistivity distribution and certain boundary condition. The inverse problem calculates the resistivity distribution given measured potential and current density distribution. In this study, we employ the Finite Element Method (FEM) to solve the forward problem, and (Tikhonov Regularization) method to find the inverse solution where it minimizes, iteratively, an objective function to obtain the update resistivity distribution equation.

Almost all of the inverse problem are ill-posed. In this case, the matrix to be inverted in the equation for calculating the update resistivity distribution is ill-conditioned. This leads to solution of that equation may not exist, or although a solution does exist, it does not stable. We adopt the well-known Tikhonov regularization technique to solve the ill-posed problem. We introduce a stabilizing or a smoothing function with a regularization parameter to the objective function, then by minimizing the value of the objective function we will obtain a regularized resistivity update equation.

This study can be seen as a continuation of [2] in the context of [7], it is worth to notes that [3] studied Tikhonov Regularization to solve EIT problem studying coefficients mathematical regularity. In [8] and [9] which is forbear of [7], they did not address in details on how to choose the regularization parameter.

## 2 Reconstruction Technique

### Calculation of Resistivity Distribution

To calculate the internal resistivity distribution of an object from boundary potential and current data, we evaluate iteratively an objective function that describes the error between the voltage response of the real object and that of the model. The objective function is defined as

$$\Pi(\rho^k) = \frac{1}{2}(\nu_e(\rho^k) - \nu_0)^T(\nu_e(\rho^k) - \nu_0) \quad (1)$$

Where  $\nu_0$  is the potential vector measured from the boundary object,  $\nu_e(\rho^k)$  is the potential vector calculated from the model of resistivity distribution. To minimize the objective function  $\Pi(\rho^k)$ , we set its derivative to zero, i.e. :

$$\Pi'(\rho^k) = \frac{\partial \Pi(\rho^k)}{\partial \rho^k} = 0, \quad (2)$$

where

$$\begin{aligned} \frac{\partial \Pi(\rho^k)}{\partial \rho^k} &= \frac{\partial \Pi(\rho^k)}{\partial \nu_e(\rho^k)} \partial \nu_e(\rho^k) \partial \rho^k \\ &= [\nu'_e(\rho^k)]^T [\nu_e(\rho^k) - \nu_0], \end{aligned}$$

where  $\nu'_e(\rho^k)$  is the Jacobian matrix. We take Taylor series expansion of  $\Pi(\rho^k)$  about the current point  $\rho^k$  then we obtain the following equation,

$$\Pi'(\rho^{k+1}) \approx \Pi'(\rho^k) + \sum_{k=2}^{\infty} \frac{\partial \Pi^n(\rho^k)}{\partial \rho^n} \frac{(\Delta \rho^k)^n}{n!}, \quad (4)$$

We keep the linear term, then we obtain the following equation

$$\Pi'(\rho^{k+1}) \approx \Pi'(\rho^k) + \Pi''(\rho^k) \Delta \rho^k = 0, \quad (5)$$

The update resistivity equation can be given by

$$\Delta \rho^k = -[\Pi''(\rho^k)]^{-1} \Pi'(\rho^k), \quad (6)$$

where :

$$\rho^{k+1} = \rho^k + \Delta \rho^k.$$

The Hessian matrix,  $\Pi''$ , is expressed as

$$\Pi''(\rho^k) = [\nu'_e(\rho^k)]^T [\nu'_e(\rho^k)] + [\nu''_e(\rho^k)] \left\{ I \times [\nu_e(\rho^k) - \nu_0] \right\}, \quad (7)$$

where  $\times$  denotes Kronecker matrix product. The second term can be omitted since it is relatively small and difficult to evaluate. Thus (7) can be expressed as :

$$\Pi''(\rho^k) = [\nu'_e(\rho^k)]^T [\nu'_e(\rho^k)] \quad (8)$$

Substituting (3) and (8) to the updation equation of resistivity distribution, we arrive at

$$\Delta\rho^k = \frac{[[\nu'_e(\rho^k)]^T [\nu'_e(\rho^k)]]^{-1}}{[\nu'_e(\rho^k)]^T [\nu_e(\rho^k) - \nu_0]}. \quad (9)$$

### Calculation of Boundary Potential Vector

For a given resistivity distribution and boundary condition, i.e., the potential and current density on the boundary, the potential distribution inside the object follows the governing equation,

$$\nabla \cdot \frac{1}{\rho} \nabla \Phi = 0, \quad \text{in } \Omega \quad (10)$$

with overdetermined boundary conditions :

$$\Phi = \Phi_0; \quad \text{on } \partial\Omega \quad (11)$$

$$\frac{1}{\rho} \frac{\partial \Phi}{\partial \eta} = \mathbf{J}_0; \quad \text{on } \partial\Omega, \quad (12)$$

where  $\Phi$  is the potential distribution within the medium,  $\Phi_0$  is the boundary potential and  $\mathbf{J}_0$  is the boundary current density, and  $\eta$  denotes normal unit vector pointed outward on the boundary. To solve the governing equation, we utilize the FEM. The region is triangulated using triangular element, under assumption that the electrical properties is homogeneous and isotropic. The FEM yields a system of linear algebraic equations :

$$\mathbf{Y} \cdot \mathbf{x} = \mathbf{I} \quad (13)$$

where  $\mathbf{Y}$  is the admittance matrix,  $\mathbf{x}$  is a voltage distribution vector, and  $\mathbf{I}$  is current vector.

The boundary potential data of the model can be calculated as follows,

$$\nu_e(\rho) = \mathbf{T}_r \text{vec}(\mathbf{x}) \quad (14)$$

where  $\mathbf{T}_r$  denotes a transformation matrix. The value of  $\nu_e(\rho)$  will be compared with the voltage measurement data in the reconstruction algorithm.

## 3 Ill-Posed Problem

### Regularization Technique

Resistivity update equation can be expressed as :

$$\mathbf{A}^T \mathbf{A} \mathbf{x} = \mathbf{A}^T \mathbf{y} \quad (15)$$

where

$$\begin{aligned} \mathbf{A} &= [\nu'_e(\rho^k)]; \\ \mathbf{x} &= \Delta\rho^k; \\ \mathbf{y} &= [\nu_e(\rho^k) - \nu_0]. \end{aligned}$$

We consider that the vector  $\mathbf{x}$  is an unknown function in a space  $\mathbf{X}$ , and a vector  $\mathbf{y}$  is in a space  $\mathbf{Y}$ . To solve the linear equation, we need to calculate the inverse of  $\mathbf{A}^T \mathbf{A}$  that contain the Jacobian. The Jacobian matrix is a function of the resistivity distribution, the injected current, the geometry object, and the boundary potential. Since the limitation of that information for  $\mathbf{y}$ , we can not obtain the approximate solution from inverting the matrix ( $\mathbf{A}^T \mathbf{A}$ ). This is because, mathematically, the operator  $\mathbf{A}$  may not belong to the mapping of  $\mathbf{X} - \mathbf{Y}$ , then  $\mathbf{x}$  may not exist, or although  $\mathbf{x}$  does exist, it does not stable, that is a small perturbation on the data, i.e., the boundary potential measurement data  $\nu_0$ , will cause a large changes in the solution of update resistivity distribution. Consequently, the problem is ill-posed, or the matrix  $\mathbf{A}^T \mathbf{A}$  is ill-conditioned, namely the ratio between the maximum eigenvalue and the minimum one is very large.

We adopt the well-known Tikhonov regularization method to solve the ill-posed problem. Here, a smoothing function  $\Lambda(\rho^k)$  is introduced to the objective function, then the (1) equation turns into :

$$\Pi(\rho^k) = \frac{1}{2}(\nu_e(\rho^k) - \nu_0)^T(\nu_e(\rho^k) - \nu_0) + \alpha\Lambda(\rho^k). \quad (16)$$

where  $\Lambda(\rho^k)$  provides the information of resistivity distribution to the objective function as a prior information,  $\alpha$  is a regularization parameter which is a positive number. We define the smoothing function as a function of the update resistivity distribution as follows,

$$\Lambda(\rho^k) = (\Delta\rho^k)^T \Sigma (\Delta\rho^k), \quad (17)$$

where  $\Sigma$  is a positive definite matrix. By minimizing the new objective function, we obtain the following resistivity update equation.

$$\Delta\rho^k = \frac{[[\nu'_e(\rho^k)]^T[\nu'_e(\rho^k)] + 4\alpha\Sigma]^{-1} [\nu'_e(\rho^k)]^T[\nu_e(\rho^k) - \nu_0]}{[\nu'_e(\rho^k)]^T[\nu_e(\rho^k) - \nu_0]}. \quad (18)$$

Observe that the equations (18) differs to (9) in term of  $[4\alpha\Sigma]$ . The matrix in (18) is more well-conditioned as  $\alpha$  is a positive number and  $\Sigma$  is a positive definite matrix.

### Parameter Selection Rule

The problem in the regularization technique is, how to determine the regularization parameter. Since when the parameter is too large, the solution will significantly be deviated from the *correct* solution, and when the parameter is too small, it does not significantly relax the problem.

1. Here, we select an initial value by trial and error technique and then we multiply it by a positive number, say  $\gamma$ , that larger than 1.0 when the value of objective function at step  $(k+1)$ -st is smaller than the step  $k$ -th. Thus if the value of objective function converges to the small value, the regularization parameter also converges to zero and the regularized update equation becomes the original equation. The algorithm for selection of the regularization can be written as follows :

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Step 1.  Set initially  $\alpha > 0, \gamma > 1.0$ 
Step 2.  IF  $\Pi(\rho^{k+1}) < \Pi(\rho^k)$  THEN
            $\alpha^{k+1} = \alpha^k / \gamma$ 
        ELSEIF  $\Pi(\rho^{k+1}) \geq \Pi(\rho^k)$  THEN
            $\alpha^{k+1} = \alpha^k \times \gamma$ 
        ENDIF

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2. For each update, we solve the updating equation (15), using non-stationary iterative Tikhonov [1], i.e. :

$$\begin{aligned} \mathbf{x}_0 &= 0 \\ \mathbf{x}_{l+1} &= (\mathbf{A}^* \mathbf{A} + \alpha I)^{-1}(\alpha \mathbf{x}_l + \mathbf{A}^* \mathbf{y}), \end{aligned}$$

where  $\mathbf{A}$  denotes Jacobian terms and  $\mathbf{y}$  as in the update equation (15). Brakhage's recommends the following, as cited in [6], on how to choose the parameter :

$$\alpha_l = \frac{\|\mathbf{A}^*(\mathbf{A}\mathbf{x}_{l-1} - \mathbf{y})\|^2}{\|\mathbf{A}\mathbf{x}_{l-1} - \mathbf{y}\|}$$

3. In [4], it cited that Engl and Gferer suggests a strategy to pick regularization parameter for linear ill-posed problem using Tikhonov Regularization, by finding roots of the following :

$$\begin{aligned} \alpha^3[(\nu(\rho^k) - \nu_0)]^t \\ [(\alpha I + \mathbf{A} \cdot \mathbf{A}^*)]^{-3}[\nu(\rho^k) - \nu_0] = \delta^2 \end{aligned}$$

where  $\delta$  denotes 'measurement error'. In the current programming, we employ automatic differentiation [5] in Newton-Raphson methods to find the roots  $\alpha$  in the formula above.

### Numerical Test

We study the algorithms with respective parameter choices using three artificially generated data sets. The first test is two regions of contrast medium representing two phase medium inside a pipe. The second test is a distribution of conductivities with rapidly oscillated towards a boundary. And the third test is artificially shows a lateral slice of chest configuration. We shows the three test cases together

with element triangulation used to generate data artificially using finite element methods. For each case, we compare the nonlinear defects error for each outer iteration :

$$d_k = \|\nu(\rho^k) - \nu_0\|.$$

In each test, the outer iteration is terminated at 11th iteration, before computational error dominates leads to unintelligible results. Furthermore in this study, we pick  $\delta = 1e-6$  as the test cases doesn't contain measurement error.

## Conclusion and Future Works.

From our numerical studies, we conclude that two a posteriori parameter choices outperform discrepancy methods. However, there is no indication which one of the two strategies is the best. From this study, it is left open to studying how to stop the iteration using a posteriori strategy as suggested by [4]. Furthermore, it would be of our interest to use the reconstruction algorithms for real data.

## References

- [1] M. Hanke & C.W. Groetsch, Nonstationary Iterated Tikhonov Regularization, *J. Optim. Th. Appl.*, **98**, 37-53, 1998.
- [2] D. Kurniadi & K.Komiya, A Regularization Technique for Inverse Electrical Impedance Tomography, In: the Regional Seminar on Computational Method and Simulation in Engineering (CMSE-97), Bandung Institute of Technology (1997)
- [3] O. Scherzer, The use of Tikhonov Regularization in the Identification of Electrical Conductivities from Overdetermined Boundary Data, *Res.Math.*, **22** (1992), 588-618.
- [4] O. Scherzer, H.W. Engl & K. Kunisch., Optimal a Posteriori Parameter Choice for Tikhonov Regularization for the Solving Nonlinear Ill-Posed Problems, *SIAM J.Numer.Anal.*, **30** (1993), 1796-1838.
- [5] W. Squire & G. Trapp., Using complex variables to estimate derivatives of real functions, *SIAM Rev.*, **40** (1998), 110-112.
- [6] S.G. Solodky., An economic approach to discretization of nonstationary iterated Tikhonov method., Technical Reports (Red Series)., Fach. Math., Uni.Kaiserslautern.
- [7] M. Vauhkonen, W.R.B. Lionheart, L.M. Heikkinen, P.J. Vauhkonen & J.P. Kaipio, A Matlab Toolbox for the EIDORS project to reconstruct two-dimensional EIT images, *Phys. Meas.*, **22**(2001), 107-111.
- [8] M. Vauhkonen, D. Vadasz, J.P. Kaipio, E. Somersalo & P.A. Karjalainen, Tikhonov Regularization and Prior Information in Electrical Impedance Tomography, *IEEE Trans Med Imaging*, **17**(1998),285-293.
- [9] M. Vauhkonen, J.P. Kaipio, E. Somersalo & P.A. Karjalainen, Electrical Impedance Tomography with Basis Constraints, *Inv Probl*, **13**(2) (1997), 523-530.

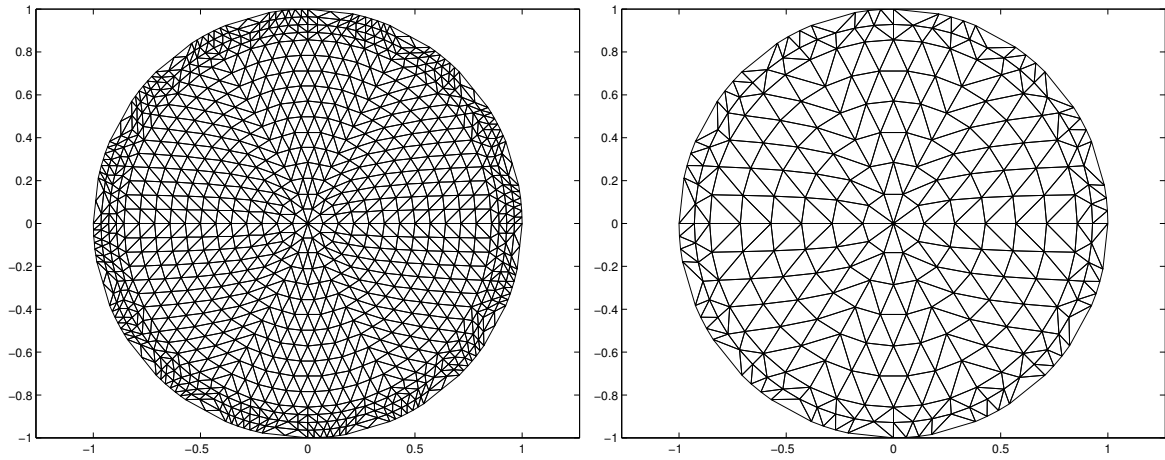


Figure 1: Element triangulation of unit circle domain. On the left is a fine finite element triangulation of the unit circle, on this triangulation sets of artificial data are generated for reconstruction later. While the coarse triangulation on the right side is the one we used for piecewise constant resistivity during reconstruction.

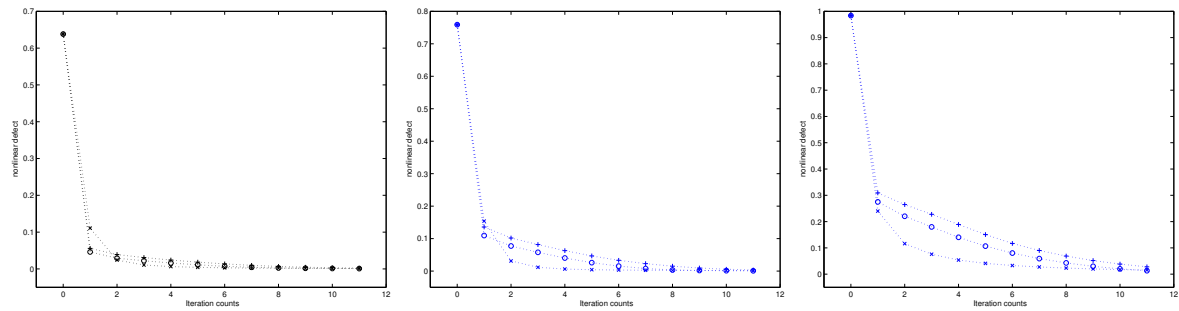


Figure 2: The nonlinear defect error vs iterations for each parameter selection on each cases. From left to right indicates cases number. On each graphs, '+' is for the discrepancy principles, 'x' is for non-stationary iterative Tikhonov, and 'o' for the third criteria.

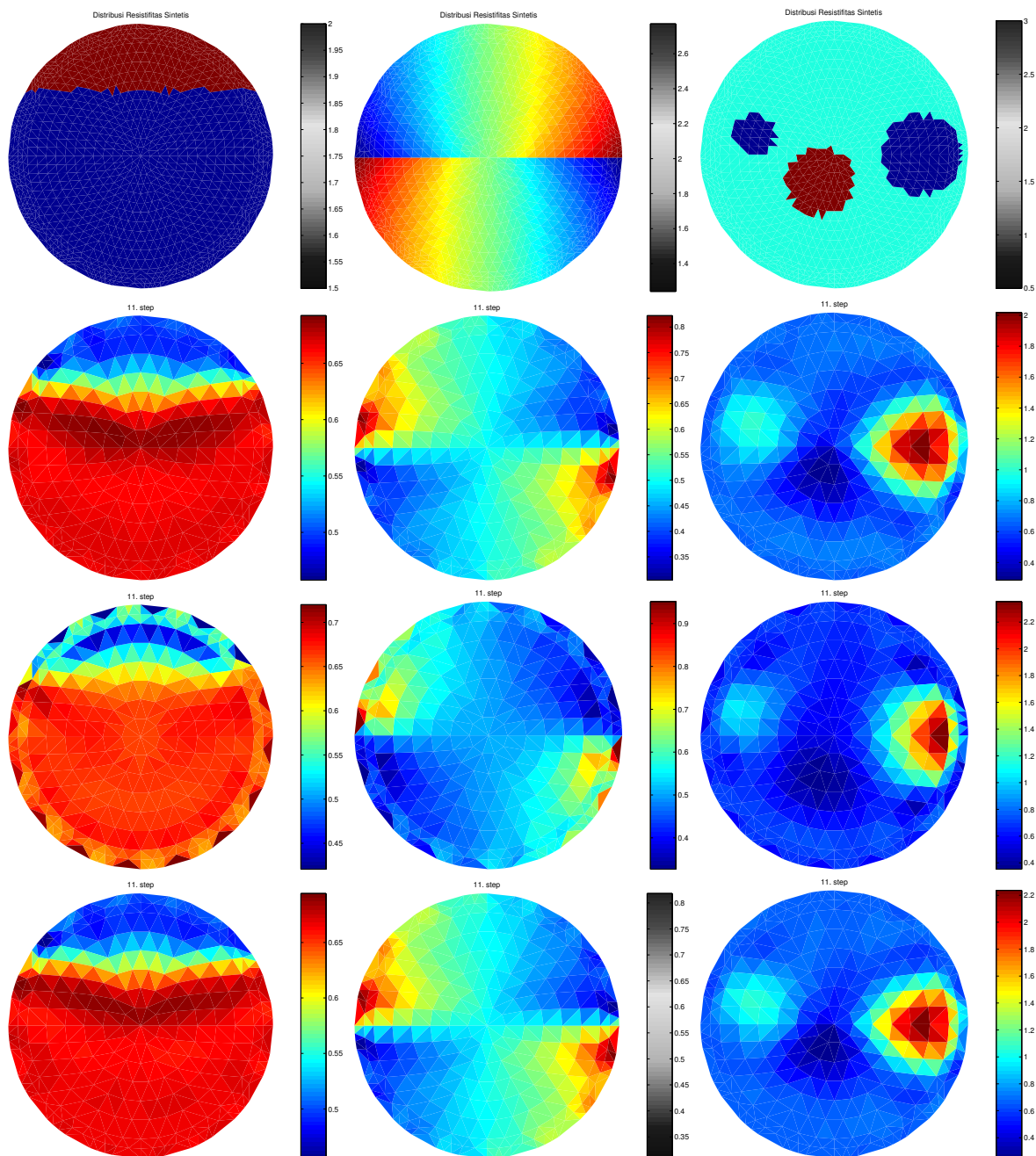


Figure 3: Top row shows resistivity distribution of an artificially generated data, and columns from left indicating each cases tested. The second row are numerical reconstruction obtained using discrepancy principle, the third row are the resulting reconstruction using non-stationary Iterative Tikhonov, and the last row the result from the third chosen criterion for a posteriori parameter choice. All reconstruction stopped at 11th iteration.