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APPROXIMATE ANALYTICAL SOLUTION OF A HIGHER ORDER WAVE EQUATION OF KdV TYPE

by

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Abstract

We consider a higher order of Korteweg-de Vries (KdV) equation which is an important water wave model. We find the approximate analytical solution of the proposed model by Exponential Homotopy Analysis Method (EHAM). By using this method, we solve the problem analytically and then compare the numerical result with exact solution. Numerical results reveal that the EHAM provides highly accurate numerical solutions for higher order KdV equation. The EHAM solution includes an auxiliary parameter. This parameter provides a convenient way of adjusting and controlling the convergence region of solution series.

1. Introduction

There are many nonlinear partial differential equations which are quite useful and applicable in engineering and physics such as the Korteweg-de

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Vries (KdV) equation. The KdV equation represents a first order approximation in the study of long wavelength, small amplitude waves of inviscid and incompressible fluids. Furthermore, if one allows the appearance of higher order terms, then more complicated wave equations can be obtained. The higher order of KdV equation is generally difficult to be solved and their exact solutions are difficult to obtain. Marinakis [11] showed that the higher order of KdV equation is integrable for a particular choice of its parameters, since in this case, it is equivalent with an integrable equation which has recently appeared in the literature. During the last century, asymptotic method [7] has often been used to obtain approximate analytical solution to these problems. These methods are typically dependent on the presence of a small parameter, consequently, asymptotic methods often fail to provide accurate results for large values of the parameters. In the recent years, much effort has been spent on this task and many significant methods have been established such as tanh-function method [5], integral bifurcation method [13] and F-expansion method [3]. An extended F-expansion method was proposed by Yomba in 2005 [2] by given more solutions of the general subequation. Using the new method, exact traveling solutions of higher order wave equation of KdV type are successfully obtained [10]. A new analytic approach named Homotopy Analysis Method (HAM) has seen rapid development. The basic idea of the HAM is to produce a succession of approximate solution that tends to the exact solution of the problem [9]. This method has been successfully applied to solve many types of nonlinear problems in dynamical fluid by many authors. Liao and Cheung [8] successfully applied HAM in fully analytical way to nonlinear waves propagation in deep water and the HAM solution in finite water depth was obtained by Tao et al. [6].

The goal of this paper has been to derive an approximate analytical solution for the higher order wave equation of the KdV type. We have achieved this goal by applying Exponential Homotopy Analysis Method (EHAM). Results are compared with the exact solution.

2. Evolution Equation

Based on the physical and asymptotic considerations, Fokas [1] derived the following generalized KdV equations:

$$\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial x}+\mu\eta\frac{\partial \eta}{\partial x}+\delta\frac{\partial^3 \eta}{\partial x^3}+\rho_1\mu^2\eta^2\frac{\partial \eta}{\partial x}+\mu\delta\left(\rho_2\eta\frac{\partial^3 \eta}{\partial x^3}+\rho_3\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2}\right)$$

$$+\rho_4\mu^3\eta^3\frac{\partial \eta}{\partial x}+\mu^2\delta\left(\rho_5\eta^2\frac{\partial^3 \eta}{\partial x^3}+\rho_6\eta\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2}+\rho_7\eta\left(\frac{\partial \eta}{\partial x}\right)^3\right)=0. \quad (1)$$

The function $\eta(x, t)$ represents the amplitude of the fluid surface with respect to its level at rest, while μ and δ characterize, respectively, the long wavelength and short amplitude of the waves, compared with the depth of the layer. Parameters ρ_i, $i = 1, 2, 3, 4, 5, 6, 7$ are free parameters. Fokas [1] assumed that $O(\delta)$ is less than $O(\mu)$. According to this assumption, we know that $O(\mu^2\delta) < O(\mu^2)$ and $O(\mu^2\delta) < O(\mu\delta)$. Neglecting two high order infinitesimal terms of $O(\mu^3, \mu^2\delta)$, equation (1) can be reduced to another high order wave equations of KdV type as follows:

$$\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial x}+\mu\eta\frac{\partial \eta}{\partial x}+\delta\frac{\partial^3 \eta}{\partial x^3}+\rho_1\mu^2\eta^2\frac{\partial \eta}{\partial x}+\mu\delta\left(\rho_2\eta\frac{\partial^3 \eta}{\partial x^3}+\rho_3\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2}\right)=0. \quad (2)$$

Equation (2) is a special case of equation (1) for $\rho_4 = \rho_5 = \rho_6 = \rho_7 = 0$. If $\rho_1 = \rho_2 = \rho_3 = 0$, then equation (2) becomes the classical KdV equation. Equation (2) is studied by many researchers and some useful results are obtained when ρ_i, $i = 1, 2, 3$ takes special values. Equation (2) was examined in [4] and it was found that it possesses solitary wave solutions which for small values of the parameters μ and δ, behave like solitons. As mentioned in [12], equation (2) is, in general, nonintegrable in the sense that some of its ordinary differential equation reductions do not possess the Painlevé property and a Lax pair does not seem to exist. However, it was still found to possess the traveling wave solution [10]:

$$\eta(x, t) = \frac{12\delta B_1(A^2 - 4B_1)\operatorname{sech}^2(\sqrt{B_1}(x - Ct - x_0))}{\mu(4\delta\rho_2 B_1 - 1)(A - 2\sqrt{B_1}\tanh(\sqrt{B_1}(x - Ct - x_0)))^2}, \tag{3}$$

when $\rho_1 = 0$ and $\rho_3 = -2\rho_2$, where $B_1 = B + \frac{1}{4}A^2$, $C = 1 + 4\delta B_1$, A and B are free constants, and x_0 being the arbitrary location of the center of the wave. Equation (3) can be written in the form of the well-known sech^2-soliton solution of the classical KdV equation:

$$\eta(x, t) = \frac{3(C-1)}{\mu(-1 + (C-1)\rho_2)}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{C-1}{\delta}}(x - Ct - x_0)\right)$$

which is exactly the one-soliton solution of the KdV if $\rho_2 = 0$ in which case, equation (2) reduces exactly to the classical KdV equation [4]. By setting $\rho_2 = 2\rho_1$, $\rho_3 = -2\rho_1$, $\rho_1 \neq 0$, the solution of (2) is

$$\eta(x, t) = \frac{12\delta B_1(A^2 - 4B_1)\operatorname{sech}^2(\sqrt{B_1}(x - Ct - x_0))}{\mu(A - 2\sqrt{B_1}\tanh(\sqrt{B_1}(x - Ct - x_0)))^2}. \tag{4}$$

3. Approximate Analytical Solution

In this section, we implement the EHAM to the higher order wave equation of KdV type. Making a transformation $\eta(x, t) = a\phi(\zeta)$, with $\zeta = x - Ct$, equation (2) can be reduced to the following ordinary differential equation:

$$(1 - C)\frac{d\phi}{d\zeta} + \mu a\phi\frac{d\phi}{d\zeta} + \delta\frac{d^3\phi}{d\zeta^3} + \rho_1\mu^2 a^2\phi^2\frac{d\phi}{d\zeta}$$

$$+ \mu\delta a\left(\rho_2\phi\frac{d^3\phi}{d\zeta^3} + \rho_3\frac{d\phi}{d\zeta}\frac{d^2\phi}{d\zeta^2}\right) = 0, \tag{5}$$

where C is the wave velocity which moves along the direction of x axis and $C \neq 0$. Suppose

$$\phi(\zeta) \sim D\exp(-\lambda\zeta) \text{ as } \zeta \to \infty, \tag{6}$$

where $\lambda > 0$ and D are constants. Substituting equation (6) into equation (5) and balancing the main terms, we have

$$\lambda = \sqrt{\frac{C-1}{\delta}}.$$

Defining $\xi = \lambda\zeta$, equation (5) becomes

$$-(C-1)\frac{d\phi}{d\xi} + (C-1)\frac{d^3\phi}{d\xi^3} + \mu a\phi\frac{d\phi}{d\xi} + \rho_1\mu^2 a^2\phi^2\frac{d\phi}{d\xi}$$

$$+ (C-1)\mu a\left(\rho_2\phi\frac{d^3\phi}{d\xi^3} + \rho_3\frac{d\phi}{d\xi}\frac{d^2\phi}{d\xi^2}\right) = 0. \tag{7}$$

Assuming the nondimensional wave elevation ϕ arrives its maximum at the origin, we have the boundary condition as follows:

$$\phi(0) = 1, \quad \frac{d\phi}{d\xi}(0) = 0, \quad \phi(\infty) = 0. \tag{8}$$

Then the solution can be expressed by the base functions

$$\{\exp(-n\xi) \mid n = 1, 2, 3, \ldots\}$$

in the form

$$\phi(\xi) = \sum_{n=1}^{\infty} b_n \exp(-n\xi), \tag{9}$$

where b_n is a coefficient to be determined. From equation (7), we define the nonlinear operator N as

$$N(\Phi, A) = -(C-1)\frac{d\phi}{d\xi} + (C-1)\frac{d^3\Phi}{d\xi^3} + \mu a\Phi\frac{d\Phi}{d\xi} + \rho_1\mu^2 a^2\Phi^2\frac{d\Phi}{d\xi}$$

$$+ (C-1)\mu a\left(\rho_2\Phi\frac{d^3\Phi}{d\xi^3} + \rho_3\frac{d\Phi}{d\xi}\frac{d^2\Phi}{d\xi^2}\right)$$

and the linear operator L is chosen as

$$L(\Phi, A) = \frac{d^3\Phi}{d\xi^3} - \frac{d\Phi}{d\xi}$$

with the property $L(C_1 \exp(-\xi) + C_2 \exp(\xi) + C_3) = 0$, where C_1, C_2 and C_3 are constants. According to the boundary condition (8), the initial guess is chosen as

$$\Phi_0(\xi) = 2\exp(-\xi) - \exp(-2\xi).$$

The EHAM is based on a continuous transform $(\Phi(\xi, p), A(p))$, as the embedding parameter p increases from 0 to 1, $(\Phi(\xi, p), A(p))$ varies from the initial guess $\Phi_0(\xi)$ to the exact solution $(\phi(\xi), a)$. To ensure this, let $h \neq 0$ denote an auxiliary parameter. We have the zeroth order deformation equation

$$(1 - p)L(\Phi(\xi, p) - \Phi_0(\xi)) = phN(\Phi(\xi, p), A(p)) \tag{10}$$

subject to the boundary conditions

$$\Phi(0, p) = 1, \quad \frac{d\Phi}{d\xi}(0, p) = 0, \quad \Phi(\infty, p) = 0.$$

Expanding $\Phi(\xi, p)$ and $A(p)$ in Taylor series with respect to p, we have

$$\Phi(\xi, p) = \Phi_0(\xi) + \sum_{m=1}^{\infty} \phi_m(\xi)p^m,$$

$$A(p) = a_0 + \sum_{m=1}^{\infty} a_m p^m,$$

where

$$\phi_m(\xi) = \frac{1}{m!}\frac{\partial^m \Phi(\xi, p)}{dp^m}|_{p=0},$$

$$a_m = \frac{1}{m!} \frac{\partial^m A(p)}{dp^m} |_{p=0}.$$

Note that equation (10) contains the auxiliary parameter h, so that $\Phi(\xi, p)$ and $A(p)$ are dependent on h. Assuming that h is so properly chosen that the series is convergent at $p = 1$, we obtain

$$\phi(\xi) = \Phi(\xi, 1) = \Phi_0(\xi) + \sum_{m=1}^{\infty} \phi_m(\xi),$$

$$a = A(1) = a_0 + \sum_{m=1}^{\infty} a_m.$$

Differentiating equations (10) m times with respect to p, then setting $p = 0$ and finally dividing them by $m!$, the mth order deformation equation is

$$L(\phi_m(\xi) - \chi_m \phi_{m-1}(\xi)) = hR_m(\vec{\phi}_m, \vec{a}_m) \tag{11}$$

subject to the boundary conditions

$$\phi_m(0) = \frac{d\phi_m(0)}{d\xi} = \phi_m(\infty) = 0, \tag{12}$$

where $\chi_m = 1$ for $m > 1$, $\chi_1 = 0$ and

$$R_m = -(C-1)\frac{d\phi_{m-1}}{d\xi} + \mu \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} a_j \phi_{i-j} \frac{d\phi_{m-i-1}}{d\xi} \right)$$

$$+ (C-1)\frac{d^3\phi_{m-1}}{d\xi^3}$$

$$+ \rho_1 \mu^2 \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} \frac{d\phi_{i-j}}{d\xi} \sum_{r=0}^{j} a_r a_{j-r} \right) \left(\sum_{t=0}^{m-1-i} \phi_t \phi_{m-1-i-t} \right)$$

$$+\mu(C-1)\left(\sum_{i=0}^{m-1}\sum_{j=0}^{i}a_j\left(\rho_2\phi_{i-j}\frac{d\phi_{m-i-1}^3}{d\xi^3}+\rho_3\frac{d\phi_{i-j}}{d\xi}\frac{d^2\phi_{m-i-1}}{d\xi^2}\right)\right).$$

The solution of equation (11) is

$$\phi_m(\xi)=C_1\exp(-\xi)+C_2\exp(\xi)+C_3+\tilde{\phi}_m(\xi),$$

where $\tilde{\phi}_m(\xi)$ is a special solution of equation (11) with the unknown terms a_{m-1}. According to the boundary condition (12) and equation (9), we have

$$C_2=C_3=0,\quad C_1=-\tilde{\phi}_m(0)\quad \text{and}\quad \tilde{\phi}_m(0)+\frac{d\tilde{\phi}_m(0)}{d\xi}=0 \qquad (13)$$

which determine a_{m-1}.

4. Result and Discussion

Suppose given the following data [4]: $\mu=1$, $\delta=0.01$, $C=1.024$, $\rho_1=0$, $\rho_2=1$, $\rho_3=-2\rho_2$. By using equations (11), (12) and (13), we successively obtain:

$$\phi_1=b_1(h)e^{-\xi}+b_2(h)e^{-2\xi}+b_3(h)e^{-3\xi}+b_4(h)e^{-4\xi}$$

$$\phi_2=d_1(h)e^{-\xi}+d_2(h)e^{-2\xi}+d_3(h)e^{-3\xi}+d_4(h)e^{-4\xi}$$

$$+d_5(h)e^{-5\xi}+d_6(h)e^{-6\xi}$$

$$a_0=0.4741833513$$

$$a_1=0.0774935614h-1.240779767\times 10^{-9}$$

$$a_2=0.0002959256h^2+0.001665576212h+1.975763962\times 10^{-10}$$

$$\vdots$$

and so on. In the same manner, the rest of the components can be obtained using the symbolic package. According to the EHAM, we can obtain the solution in a series form as follows:

$$\phi(\xi) = \Phi_0(\xi) + \phi_1(\xi) + \phi_2(\xi) + \phi_3(\xi) + \phi_4(\xi) + \cdots,$$

$$a = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots. \tag{14}$$

The above series (14) contains the auxiliary parameter h which influences the convergent region. For different values of h, a converges to the same value - the approximation of the exact solution. It can be seen in Figure 1, the nearly horizontal line segments of $a - h$ curves correspond to the convergence regions of the h values. The valid region of h in this case is $-2 < h < -1/2$ as shown in Figure 1. The convergence region enlarges as more high order terms are included in the series. Based on the above arguments, the auxiliary parameter is chosen as $h = -0.9$ for all the EHAM solutions presented in this section.

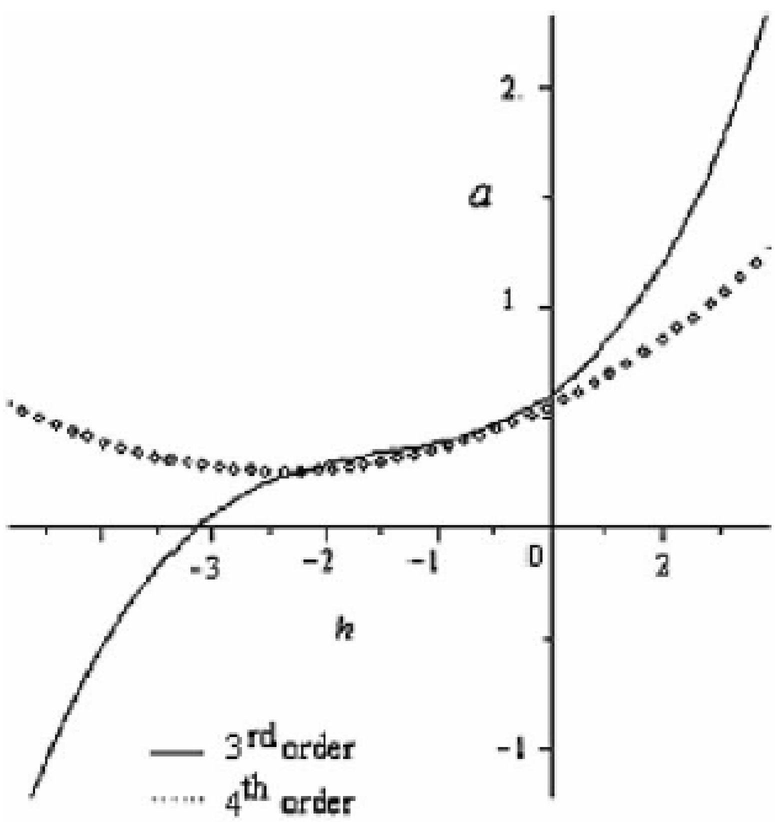

Figure 1. The h-curves of a by EHAM.

The comparison of the EHAM solution and the exact solution is shown in Figure 2. It can be seen in Figure 2 that the present EHAM solution is almost identical with the exact solution. There exists a very good agreement between EHAM solution and exact solution.

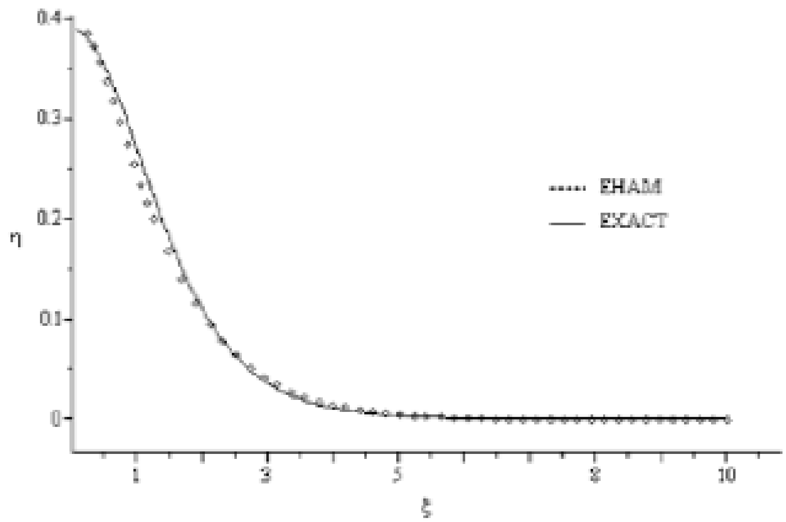

Figure 2. Comparison of the exact solution with the 6th order EHAM solution of η.

5. Conclusions

In this paper, EHAM has been successfully applied to find the approximation analytical solution of the higher order wave equation of Korteweg-de Vries. The convergence region is controlled by the non-zero parameter, providing us a simple way to adjust convergence. The present method holds promise in providing traveling wave solution for more complicated wave equations.

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