Recent Development on Estimation of the Mean Function of a Compound Cyclic Poisson Process

I Wayan Mangku[1]*

[1]Department of Mathematics, Faculty of Mathematics and Natural Sciences, IPB University

*Corresponding email: wayanma@apps.ipb.ac.id

Abstract

Compound cyclic Poisson process is a special form of compound inhomogeneous Poisson process, which has many applications in applied sciences. The objective of this paper is to survey some recent development on estimation of the mean function of a compound cyclic Poisson process. The presented results will include formulation of the estimator, consistency, asymptotic approximations to its bias and variance, asymptotic normality, and a confidence interval for the mean function. We will also show that, in order to have asymptotic bias and asymptotic variance of the estimator, it is needed to modify the estimator. Furthermore, in order to have asymptotic normality of the estimator, it is needed to rewrite the estimator as a sum of independent components. Some simulation results on distribution of the estimator and some applications of this process will also be presented.

Keywords: Asymptotic normality, compound Poisson process, confidence interval, consistency, mean function.

1. Introduction

A process $\{Y(t),\, t \geq 0\}$ is called *a compound Poisson process* if this process can be expressed as

$$Y(t) = \sum_{i=1}^{N(t)} X_i,\ t \geq 0,$$

with $\{N(t),\, t \geq 0\}$ is a Poisson process with rate λ and $X_1, X_2, \ldots$ is a sequence of i.i.d random variables having distribution function F, and also independnet to the process $\{N(t)\}$. This kind of process has wide applications in applied sciences. Some examples are as follows.

If $X_1, X_2, \ldots$ is a sequence of claims by customers and $\{N(t)\}$ denotes the number of claims, then $\{Y(t),\, t \geq 0\}$ represents the total claims by customers in the period $[0, t]$. If $X_1, X_2, \ldots$ denote the amount of money spent by the first, second, and so on customers, and $\{N(t)\}$ denotes the number of customers, then $\{Y(t),\, t \geq 0\}$ represents the total amount of money spent by all customers in the period $[0, t]$.

The Problem arise is as follows. The available theory (model) in the literature usually assume that $\{N(t)\}$ is a homogeneous Poisson process, that is a process with costant intensity (rate). In most real applications, the intensity of the procees is not constant, but it is a function of time t.

Hence it is needed more suitable model to be used, that is a compound nonhomogeneous Poisson process. One special case of compound nonhomogeneous Poisson process is the compound cyclic Poisson process. In this model, the compnent $\{N(t)\}$ is a cyclic Poisson process, that is a Poisson process having cyclic (periodic) intensity function.

2. Compound Cyclic Poisson Process

The process

$$Y(t) = \sum_{i=1}^{N(t)} X_i,\ t \geq 0,$$

is called a compound cyclic Poisson process if $\{N(t), t > 0\}$ is a cyclic Poisson process, that is the Poisson process having cyclic (periodic) intensity function, say $\lambda(s)$. Since it is very difficult to estimate distribution of this process, some researchers interested to estimate two aspects of this process, namely its mean function and its variance function. In this paper, we only discuss estimation of the mean function of compound cyclic Poisson process. Some results on estimation of the variance function of compound cyclic Poisson process can be found in Makhmudah *et al.* 2016.

The mean function of this process is defined as follows

$$\psi(t) = E\,[Y(t)] = E[N(t)]E[\geq X_i] = \Lambda(t)\mu,$$

with $\mu = E(X_i)$ and $\Lambda(t) = \int_0^t \lambda(s)\, ds$.

To employ periodicity of the intensity function of this process, we do the following. Let $t_r = t - \left\lfloor \frac{t}{\tau} \right\rfloor \tau$, where $\lfloor x \rfloor$ denotes the biggest integer that less than or equal to x, $x \in \mathbb{R}$, and $k_{t,\tau} = \left\lfloor \frac{t}{\tau} \right\rfloor$. Then for any real number $t \geq 0$ we can write $t = k_{t,\tau}\tau + t_r$ with $0 \leq t_r < \tau$. Let $\theta = \frac{1}{\tau}\int_0^\tau \lambda(s)\, ds$, that is the global intensity of the process $\{N(t), t \geq 0\}$ and it is assumed that $\theta > 0$.

By using the above notations, $\Lambda(t)$ can be written as

$$\Lambda(t) = \int_0^t \lambda(s) ds = k_{t,\tau}\tau\theta + \Lambda(t_r),$$

and the mean function of $Y(t)$ now can be written as

$$\psi(t) = \left(k_{t,\tau}\tau\theta + \Lambda(t_r)\right)\mu.$$

2. Formulation of the Estimator and Its Consistency

Let for some $\omega \in \Omega$, a single realization $N(\omega)$ of the process $\{N(t), t \geq 0\}$ that defined on a probability space $(\Omega, \mathcal{F}, P)$ is observed on a bounded interval $[0, n]$. Suppose that for each data point in the observed realization $N(\omega) \cap [0, n]$, say i-th data point, $i = 1, 2, \ldots, N[0, n]$, its corresponding random variable X_i is also observed. An estimator of the mean function has been proposed in Ruhiyat *et al.* 2013 as follows

$$\hat{\psi}_n(t) = \left(k_{t,\tau}\tau\hat{\theta}_n + \hat{\Lambda}_n(t_r)\right)\hat{\mu}_n,$$

with

$$\hat{\theta}_n = \frac{N([0,n])}{n},$$

$$\hat{\Lambda}_n(t_r) = \frac{\tau}{n}\sum_{k=1}^{k_{n,\tau}} N([k\tau, k\tau + t_r]),$$

and

$$\hat{\mu}_n = \frac{1}{N([0,n])}\sum_{i=1}^{N([0,1])} X_i.$$

Note that, if $N[0, n] = 0$, we define $\hat{\mu}_n = 0$, which implies $\hat{\psi}_n(t) = 0$. In Ruhiyat *et al.* 2013, the estimator $\hat{\psi}_n(t)$ has been proved to be weakly and strongly consistent estimator for the mean function $\psi(t)$.

3. Asymptotic Approximations to the Bias and Variance

In order to have asymptotic approximations to the bias and variance, it is needed to modify our estimator as follows (Mangku *et al.* 2013)

$$\hat{\psi}_{2,n}(t) = \left(k_{t,\tau}\tau\hat{\theta}_{2,n} + \hat{\Lambda}_{2,n}(t_r)\right)\hat{\mu}_n,$$

with

$$\hat{\theta}_{2,n} = \frac{1}{k_{n,\tau}\tau}\sum_{k=0}^{k_{n,\tau-1}} N([k\tau, k\tau + \tau]),$$

$$\hat{\Lambda}_{2,n}(t_r) = \frac{1}{k_{n,\tau}}\sum_{k=0}^{k_{n,\tau-1}} N([k\tau, k\tau + t_r]),$$

and

$$\hat{\mu}_n = \frac{1}{N([0,n])}\sum_{i=1}^{N([0,1])} X_i.$$

In Mangku *et al.* 2013 has been proved the following two results.
If the intensity function λ is periodic and locally integrable, then we have

$$\mathrm{E}(\hat{\psi}_{2,n}(t)) = \psi(t) - \frac{\psi(t)}{e^{-n\theta}} + o(e^{-n})$$

and

$$\mathrm{Var}(\hat{\psi}_{2,n}(t)) = \frac{\mu^2\tau}{n}\left(k_{t,\tau}^2 + \Lambda(t_r)(1 + 2k_{t,\tau})\right) + \frac{\sigma^2}{\theta n}\left(k_{t,\tau}\theta\tau + \Lambda(t_r)\right)^2 + O(n^{-2})$$

as $n \to \infty$. We note that, since the bias of $\hat{\psi}_{2,n}(t)$ is of smaller order than $O(n^{-2})$, we have that the rate of decrease of $\mathrm{MSE}(\hat{\psi}_{2,n}(t))$ is of order $O(n^{-1})$, as $n \to \infty$.

5. Asymptotic Normality of the Modified Estimator

In order to have an asymptotic normality of our estimator, further modification of the estimator is needed. Recall our estimator

$$\hat{\psi}_{2,n}(t) = \left(k_{t,\tau}\tau\hat{\theta}_{2,n} + \hat{\Lambda}_{2,n}(t_r)\right)\hat{\mu}_n.$$

The problem of this estimator is that $\hat{\theta}_{2,n}$ and $\hat{\Lambda}_{2,n}(t_r)$ are not independent. Our idea is to have a new estimator which is a sum of weighted two independent random variables multiplied by $\hat{\mu}_n$. First note that $\theta\tau = \int_0^\tau \lambda(s)ds$ can be written as $\Lambda(t_r) + \Lambda^c(t_r)$, where

$$\Lambda(t_r) = \int_0^{t_r} \lambda(s)ds \text{ and } \Lambda^c(t_r) = \int_{t_r}^{\tau} \lambda(s)ds.$$

Hence for any $t \geq 0$, we have

$$\Lambda(t) = \left(1 + k_{t,\tau}\right)\Lambda(t_r) + k_{t,\tau}\Lambda^c(t_r),$$

and the mean function of $Y(t)$ now can be written as

$$\psi(t) = \left(\left(1 + k_{t,\tau}\right)\Lambda(t_r) + k_{t,\tau}\Lambda^c(t_r)\right)\mu.$$

An Estimator of $\Lambda(t)$ is given by

$$\hat{\Lambda}_{2,n}(t) = \left(1 + k_{t,\tau}\right)\hat{\Lambda}_{2,n}(t_r) + k_{t,\tau}\hat{\Lambda}_{2,n}{}^{c}(t_r),$$

with

$$\hat{\Lambda}_{2,n}(t_r) = \frac{1}{k_{n,\tau}}\sum_{k=0}^{k_{n,\tau}-1} N([k\tau, k\tau + t_r]),$$

and

$$\widehat{\Lambda}_{2,n}{}^{c}(t_r) = \frac{1}{k_{n,\tau}} \sum_{k=0}^{k_{n,\tau}-1} N([k\tau + t_r, k\tau + \tau]).$$

Note that $\widehat{\Lambda}_{2,n}(t_r)$ and $\widehat{\Lambda}_{2,n}{}^{c}(t_r)$ are independent random variables.

Finally we have our new estimator for the mean function $\psi(t)$ as follows

$$\hat{\psi}_{3,n}(t) = \left(\left(1 + k_{t,\tau}\right)\widehat{\Lambda}_{2,n}(t_r) + k_{t,\tau}\widehat{\Lambda}_{2,n}{}^{c}(t_r)\right)\hat{\mu}_n.$$

Asymptotic normality of this estimator is given as follows. If the intensity function λ is periodic and locally integrable, then we have (Adriani, 2019)

$$\sqrt{n}\left(\hat{\psi}_{3,n}(t) - \psi(t)\right) \xrightarrow{d} \text{Normal}(0,\ V(t)),$$

or

$$\frac{\sqrt{n}}{\sqrt{V(t)}}\left(\hat{\psi}_{3,n}(t) - \psi(t)\right) \xrightarrow{d} \text{Normal}(0,\ 1)$$

as $n \to \infty$, where

$$V(t) = \left(1 + k_{t,\tau}\right)^2 \Lambda(t_r)\tau\mu^2 + k_{t,\tau}{}^2\Lambda^c(t_r)\tau\mu^2 + \frac{\sigma^2\Lambda(t)^2}{\theta}.$$

6. Confidence Interval

Based on a studentize version of our asymptotic normality, we have the following confidence interval for the mean function $\psi(t)$. For a given significant level α, with $0 < \alpha < 1$, a confidence interval for the mean function $\psi(t)$ is given by

$$I_{\psi,n} = \left(\hat{\psi}_n(t) \pm \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\sqrt{\frac{\hat{V}_n(t)}{n}}\right)$$

where

$$\hat{V}_n(t) = \left(1 + k_{t,\tau}\right)^2 \widehat{\Lambda}_{2,n}(t_r)\tau\hat{\mu}_n{}^2 + k_{t,\tau}{}^2\widehat{\Lambda}_{2,n}{}^{c}(t_r)\tau\hat{\mu}_n{}^2 + \frac{\hat{\sigma}_n{}^2\hat{\Lambda}_n(t)^2}{\hat{\theta}_n}.$$

The following result has been prove in Fithry *et al.* (2022). For the confidence interval $I_{\psi,n}$ for $\psi(t)$ we have that

$$P(\psi(t) \in I_{\psi,n}) \to 1 - \alpha,$$

as $n \to \infty$.

7. Conclusion

A series of research in estimation of the mean function of a compound cyclic Poisson process has been done. The first result is formulation of a consistent estimator for the mean function of a compound cyclic Poisson process. After a little modification of this estimator, asymptotic approximations to the bias and variance of the modified estimator have been obtained. By further modification of the estimator, an asymptotic normality has been established. Finally, a confidence interval for the mean function a compound cyclic Poisson process has been formulated.

References

[1] Adriani IR. 2019. *Asmptotic normalities of estimators for the mean and the variance functions of a compound cyclic Poisson process* [thesis]. Bogor (ID): IPB University.

[2] Fithry A, Mangku IW, Purnaba IGP. 2022. Interval kepercayaan untuk fungsi nilai harapan dan fungsi ragam proses Poisson periodik majemuk. *Jambura Journal of Mathematics*, 4(1), 152-166.

[3] Makhmudah FI, Mangku IW, Sumarno H. 2016. Estimating the variance function of a compound cyclic Poisson process. *Far East Journal of Mathematical Sciences*, 100(6):911-922.

[4] Mangku IW, Ruhiyat, Purnaba IGP. 2013. Statistical properties of an estimator for the mean function of a compound cyclic Poisson process. *Far East Journal of Mathematical Science*, 82, No.2, 227-237.

[5] Ruhiyat, Mangku IW, Purnaba IGP. 2013. Consistent estimation of the mean function of a compound cyclic Poisson process. *Far East Journal of Mathematical Science*, 77, No.2, 183-194.